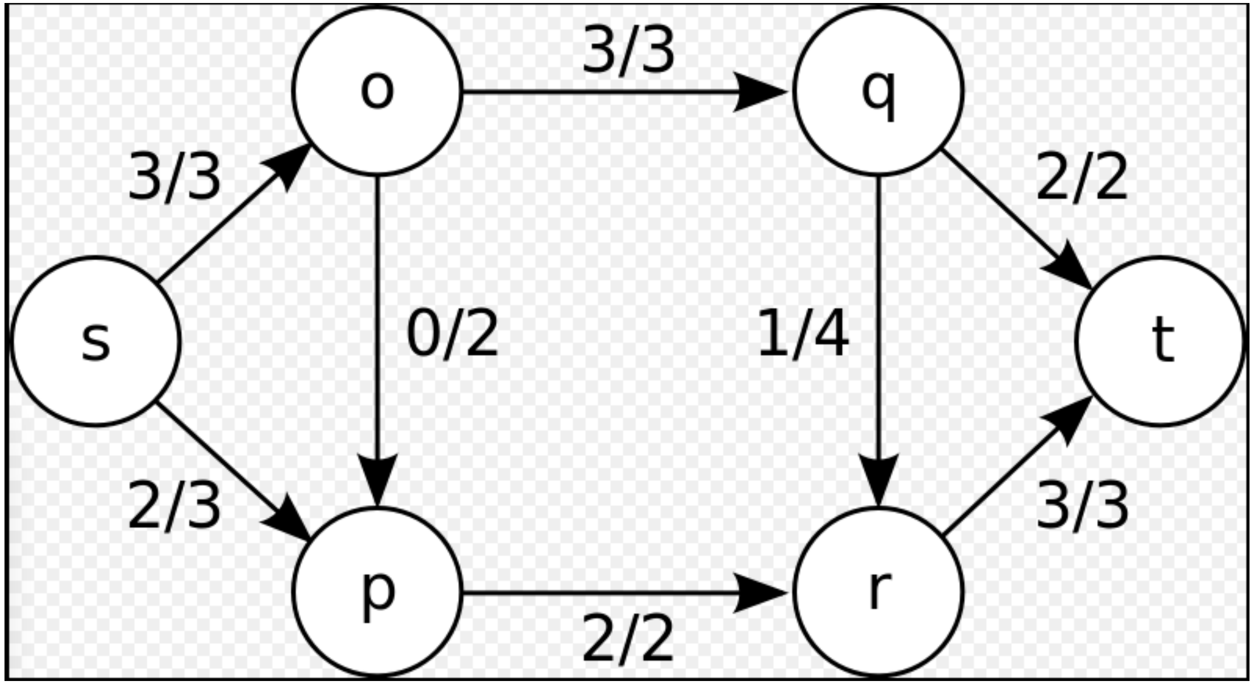
Parallel Graph Algorithms
(continued)

MaxFlow



- A flow network G=(V,E): a directed graph, where each edge $(u,v) \in E$ has a nonnegative capacity $c(u,v)>=0$.
- If $(u,v) \notin E$, we assume that $c(u,v)=0$.
- Two distinct vertices : **source s and sink t.**

Find f: E -> R, such that

- Capacity constraint: For all u,v $\in$ V,

  we require $f(u,v) \le c(u,v)$

- Flow conservation: For all u $\in V \setminus \{s,t\}$,

  we require $$\sum_{e.in.v} f(e) = \sum_{e.out.v} f(e)$$

- Maximize $$|f| = \sum_{v \in V} f(s,v)$$

A Long History

Initially defined by Ford and Fulkerson (1956)

Date	Discoverer	Running time
1969	Edmonds and Karp	$O(nm^2)$
1970	Dinic	$O(n^2m)$
1974	Karzanov	$O(n^3)$
1977	Cherkasky	$O(n^2m^{1/2})$
1978	Malhotra, Pramodh Kumar, and Maheshwari	$O(n^3)$
1978	Galil	$O(n^{5/3}m^{2/3})$
1978	Galil and Naamad; Shiloach	$O(nm(\log n)^2)$
1980	Sleator and Tarjan	$O(nm \log n)$
1982	Shiloach and Vishkin	$O(n^3)$
1983	Gabow	$O(nm \log U)$
1984	Tarjan	$O(n^3)$
1985	Goldberg	$O(n^3)$
1986	Goldberg and Tarjan	$O(nm \log(n^2/m))$
1986	Ahuja and Orlin	$O(nm + n^2 \log U)$

MaxFlow for sparse digraphs with m edges and integer capacities between 1 and C

1997	length function	$O(m^{3/2} \log m \log C)$	Goldberg-Rao
2012	compact network	$O(m^2 / \log m)$	Orlin
?	?	$O(m)$	?

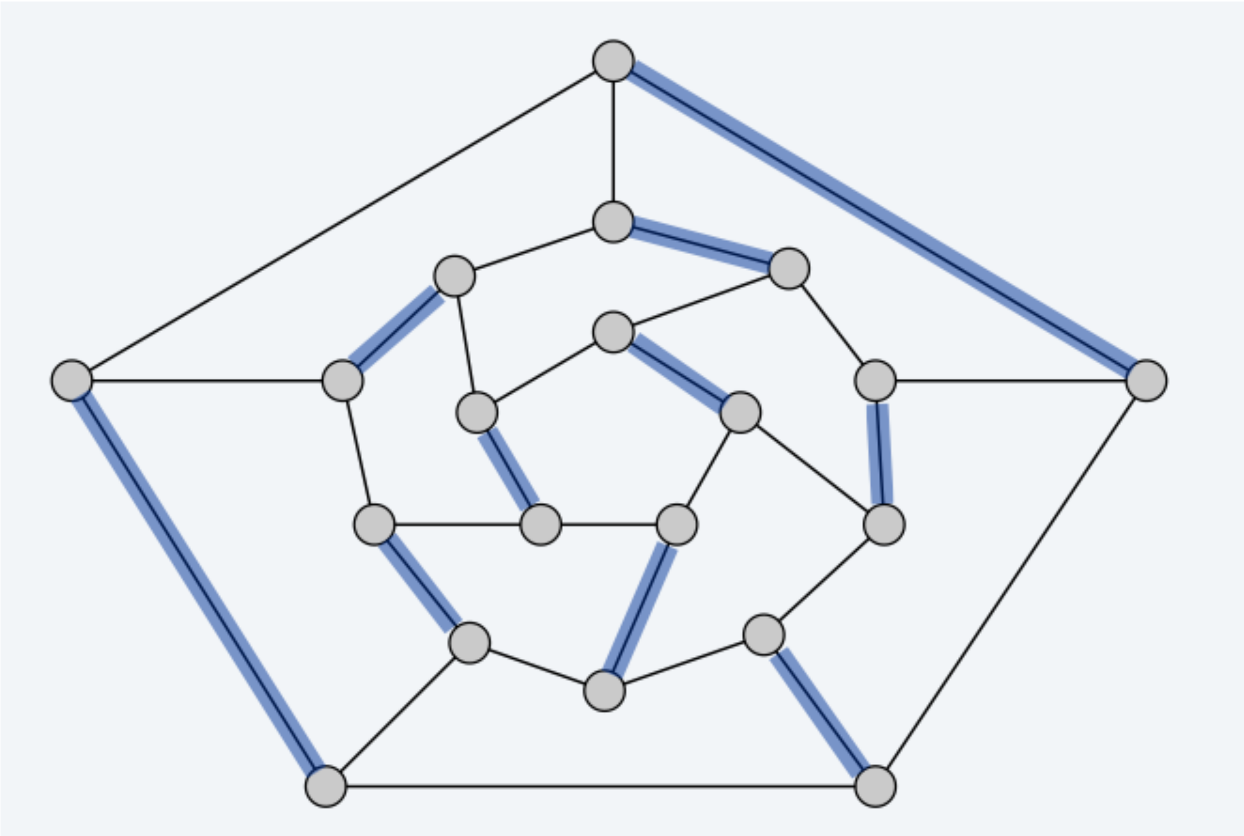
Applications

- Data mining.
- Open-pit mining.
- Bipartite matching.
- Network reliability.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Distributed computing.
- Security of statistical data.
- Egalitarian stable matching.
- Network intrusion detection.
- Multi-camera scene reconstruction.
- Sensor placement for homeland security.
- Many, many, more.

Example: Matching

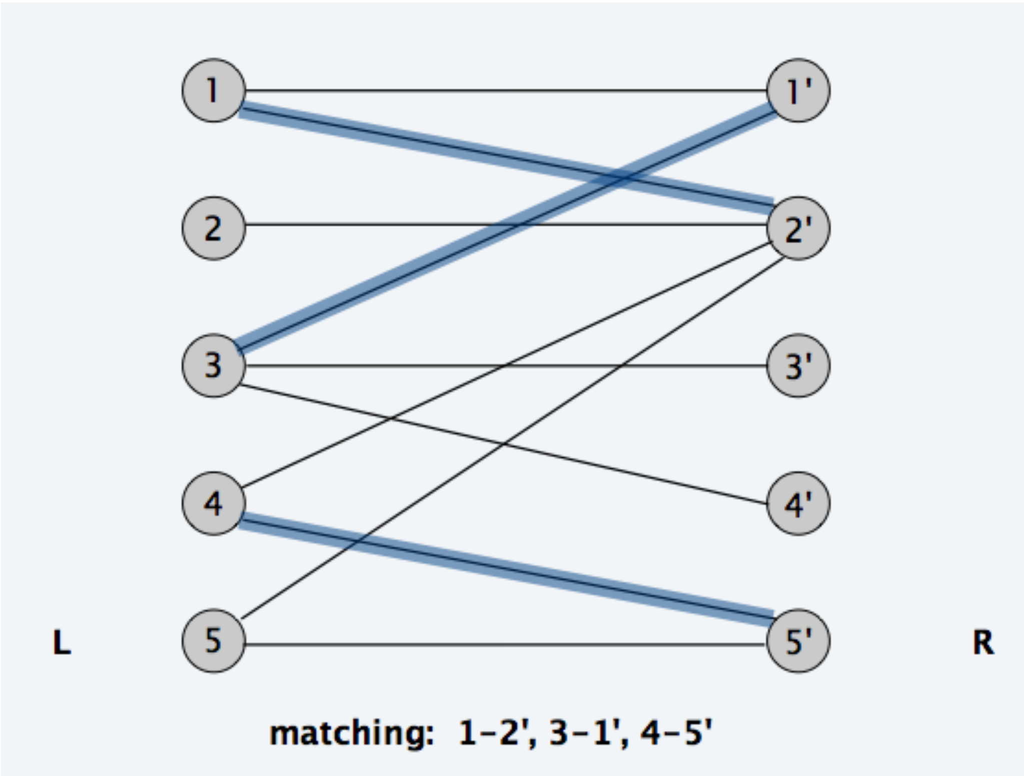
Given an undirected graph G = (V, E) a subset of edges M $\subseteq$ E is a matching if each node appears in at most one edge in M.

Max matching: Given a graph, find a max cardinality matching.



Bipartite Matching

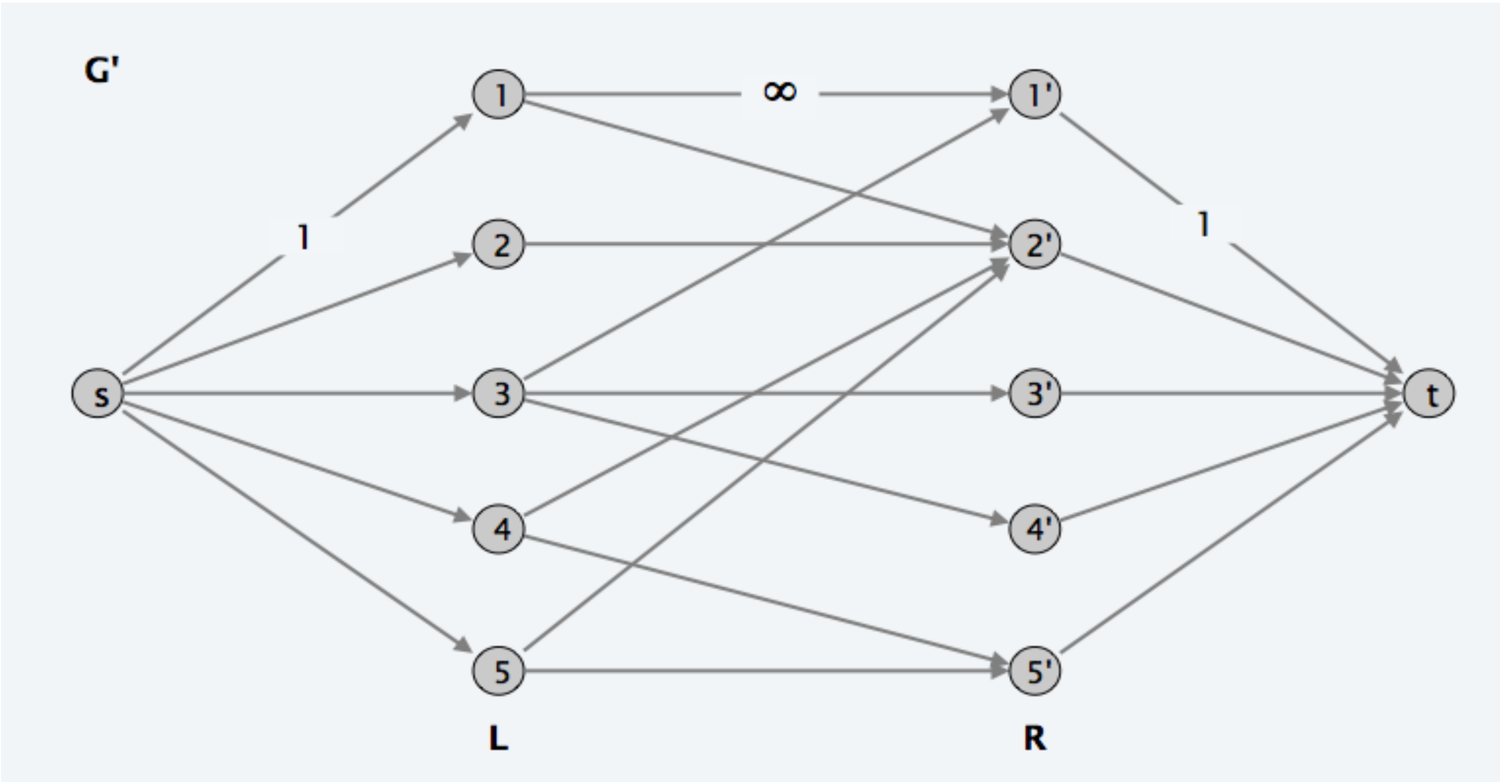
A graph G is bipartite if the nodes can be partitioned into two subsets L and R such that **every** edge connects a node in L to one in R



Note that nodes 2, 5, 3’ and 4’ are **not covered**

Bipartite Matching: Maxflow Formulation

- Create digraph G'= (L ∪ R ∪ {s, t}, E').
- Direct all edges from L to R, and assign infinite capacity.
- Add source s, and unit capacity edges from s to each node in L.
- Add sink t, and unit capacity edges from each node in R to t.



Solving MaxFlow: The Ford-Fulkerson method

The Ford-Fulkerson method depends on three important ideas that transcend the method and are relevant to many flow algorithms and problems: residual networks, augmenting paths, and cuts. These ideas are essential to the important max-flow min-cut theorem, which characterizes the value of maximum flow in terms of cuts of the flow network.

FORD-FULKERSON-METHOD(G,s,t)

initialize flow *f* to *0*

while there exists an *augmenting* path *p*

    do *augment* flow *f* along *p*

return *f*

Augmenting Paths

- Given a flow network and a flow, the **residual network** consists of edges that can admit more net flow.
- The amount of additional net flow from *u* to *v* before exceeding the capacity $c(u,v)$ is the **residual capacity** of (u,v), given by:

$$c_f(u,v)=c(u,v) - f(u,v)$$

and in the other direction:

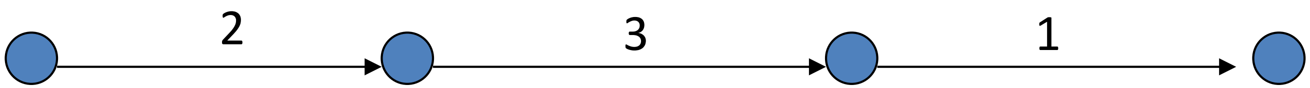
$$c_f(v, u)=c(v, u) + f(u, v).$$

- If f is a flow in G and f' is a flow in the residual network G_f then $f + f'$ is also a valid flow in G

Augmenting Paths

- Given a flow network G=(V,E) and a flow f, an augmenting path is a simple path from s to t in the residual network G_f .
- Residual capacity of p : the maximum amount of net flow that we can ship along the edges of an augmenting path p, i.e.,

$$c_f(p)=\min\{c_f(u,v):(u,v) \text{ is on } p\}.$$



The residual capacity is 1

The basic Ford-Fulkerson algorithm

FORD-FULKERSON(G,s,t)

for each edge $(u,v) \in E[G]$

  do $f[u,v] = 0$; $c_f(u,v) = c(u,v)$;

    $f[v,u] = 0$; $c_f(v,u) = 0$

while there exists a path p from s to t in the residual network G_f

  do $c_f(\mathrm{p}) = \min\{c_f(x,y)$: (x,y) is in p$\}$

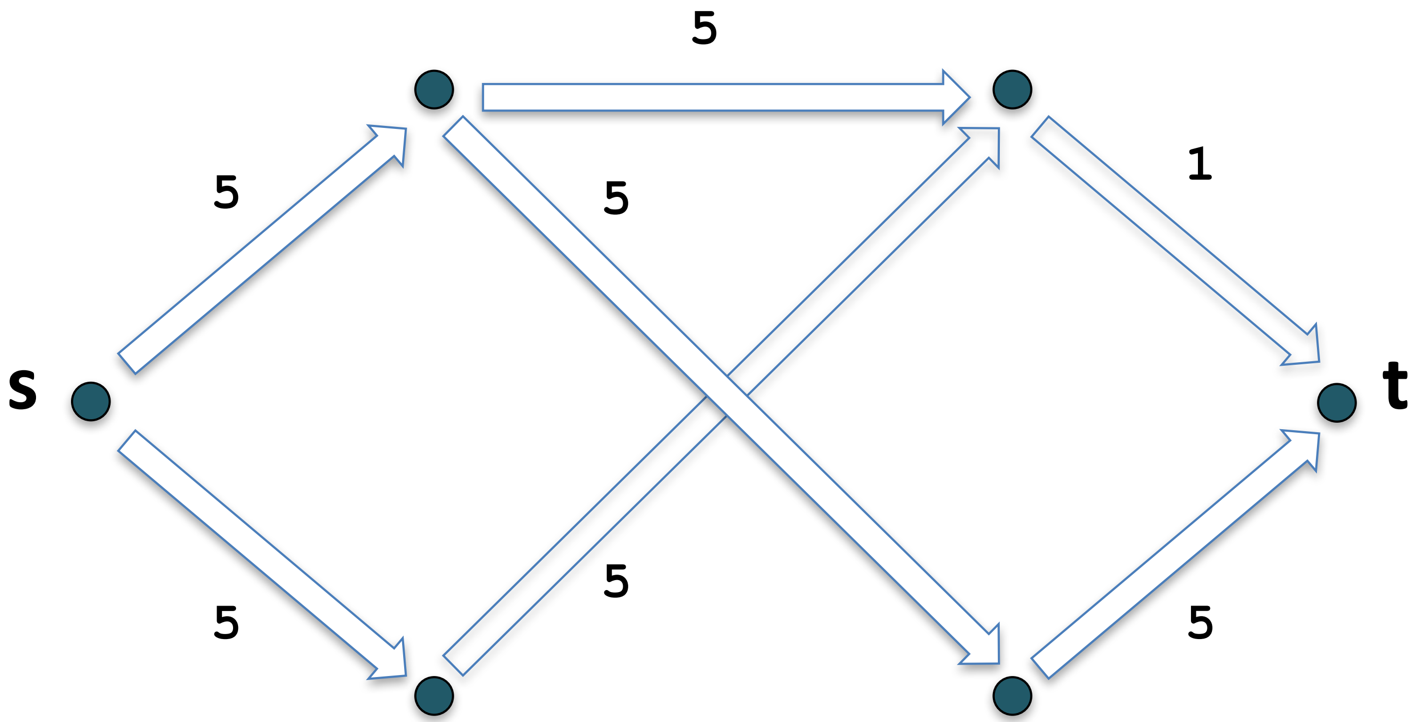
  for each edge (x,y) in p

    do $f[x,y] = f[x,y] + c_f(\mathrm{p})$;

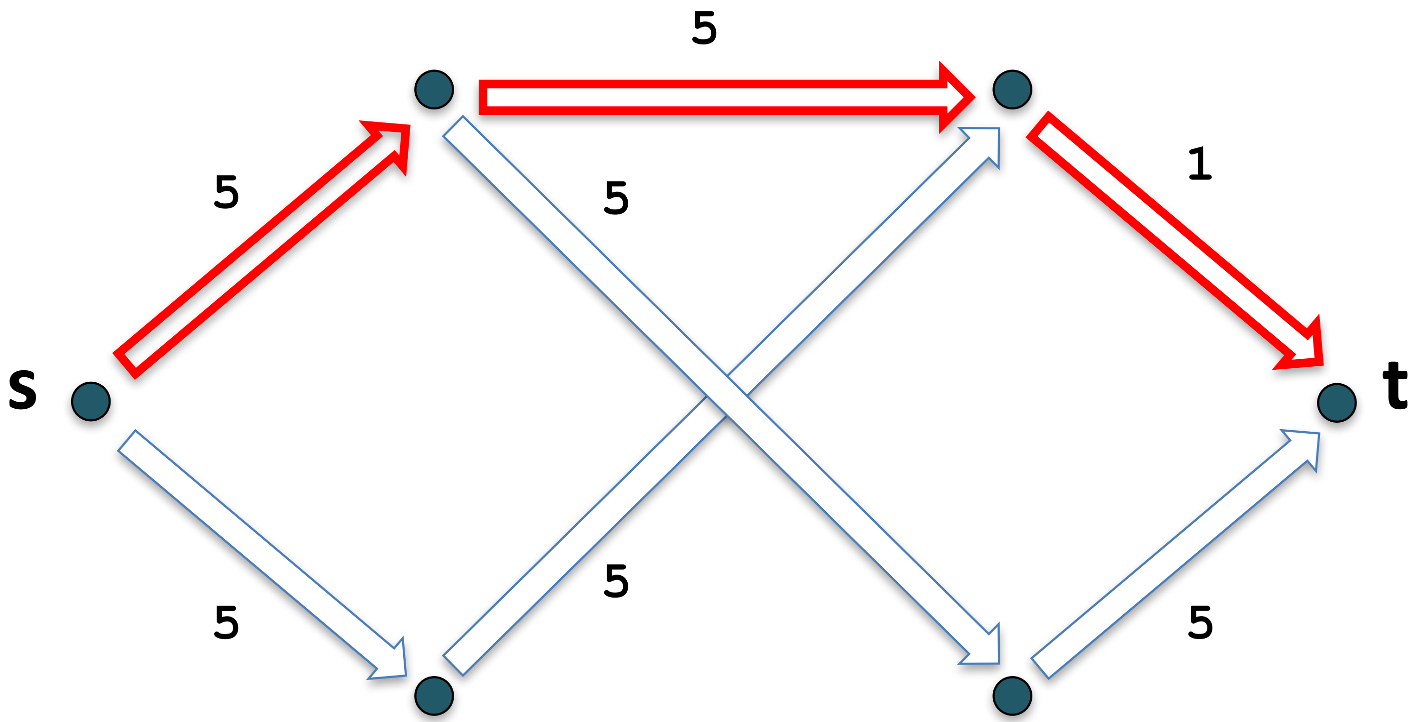
     $c_f(x,y) = c(x,y) - c_f(\mathrm{p})$;

     $c_f(y,x) = c(y,x) + c_f(\mathrm{p})$;

Why for every edge (u,v) : (v,u)?

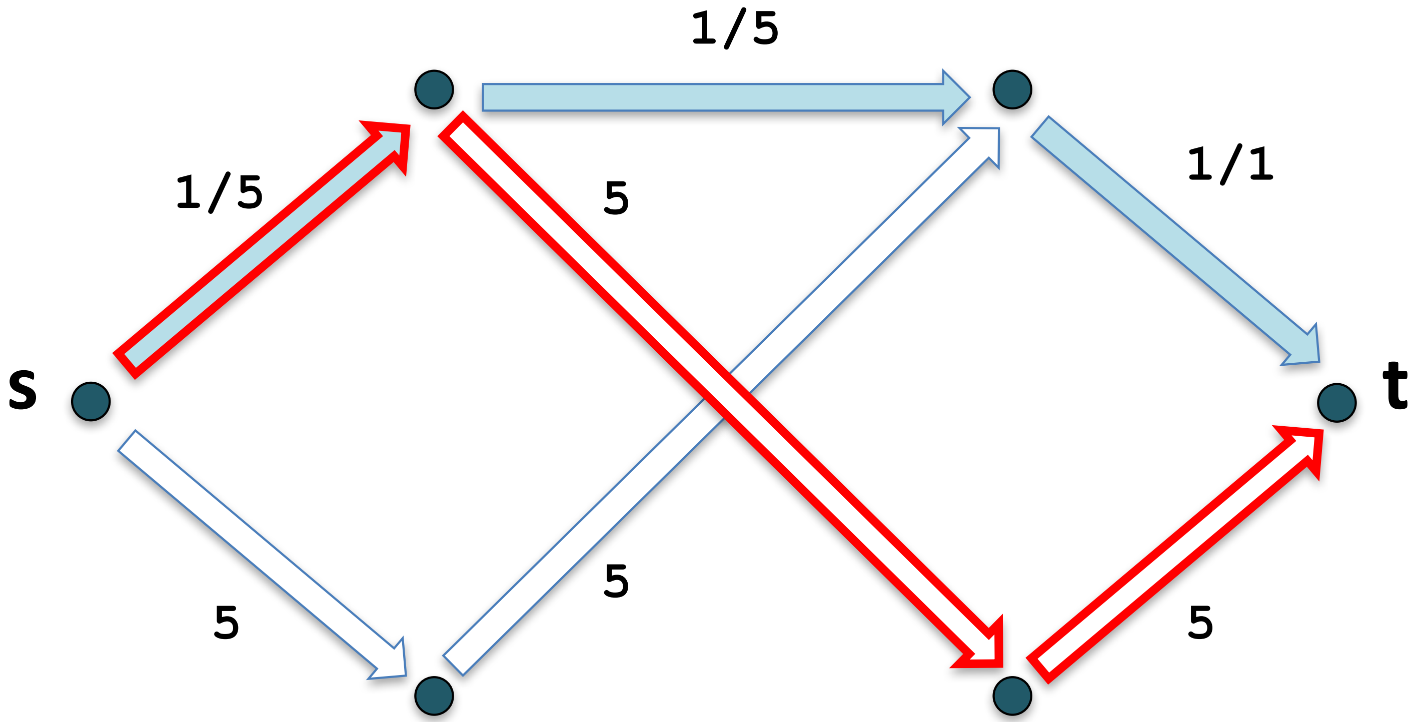


Why for every edge (*u*,*v*) : (*v*,*u*)?



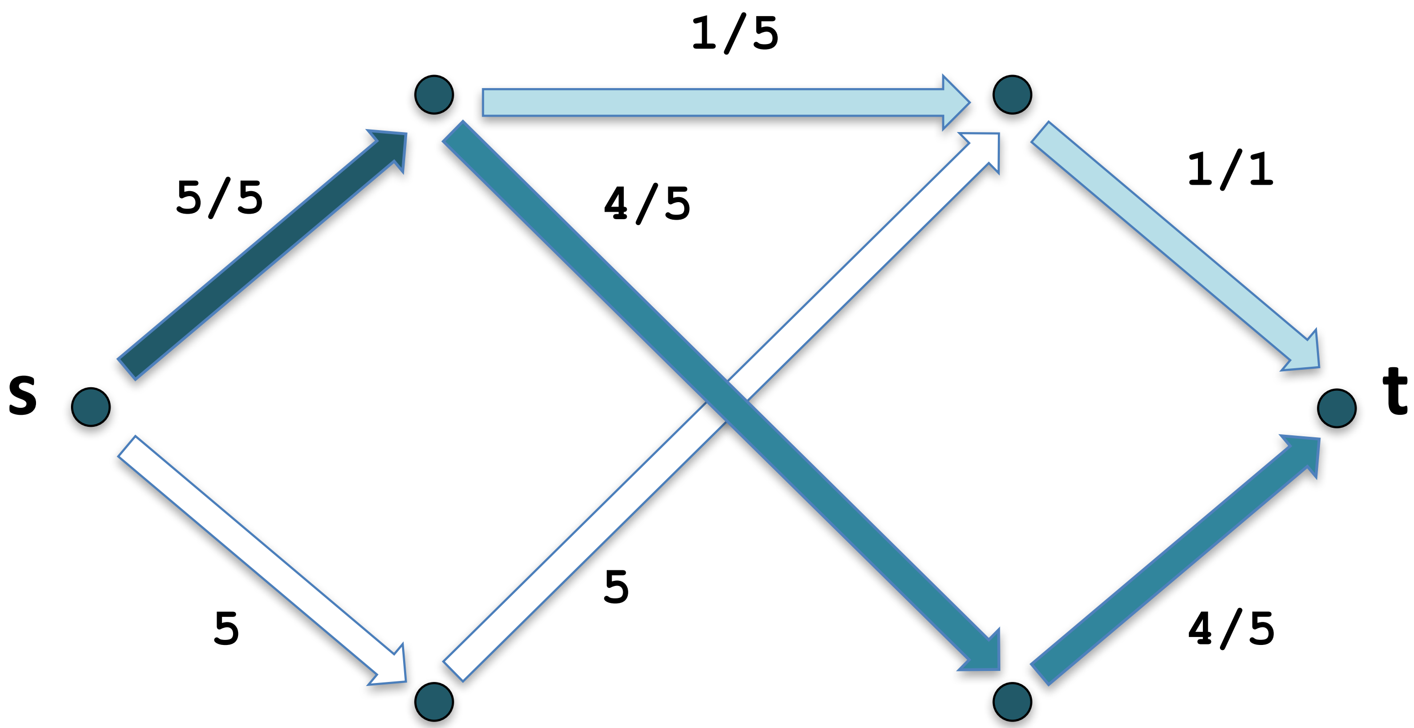
Augmented path with residual capacity = min(5,5,1) = 1

Why for every edge (u,v) : (v,u)?



Augmented path with residual capacity = min(4,5,5) = 4

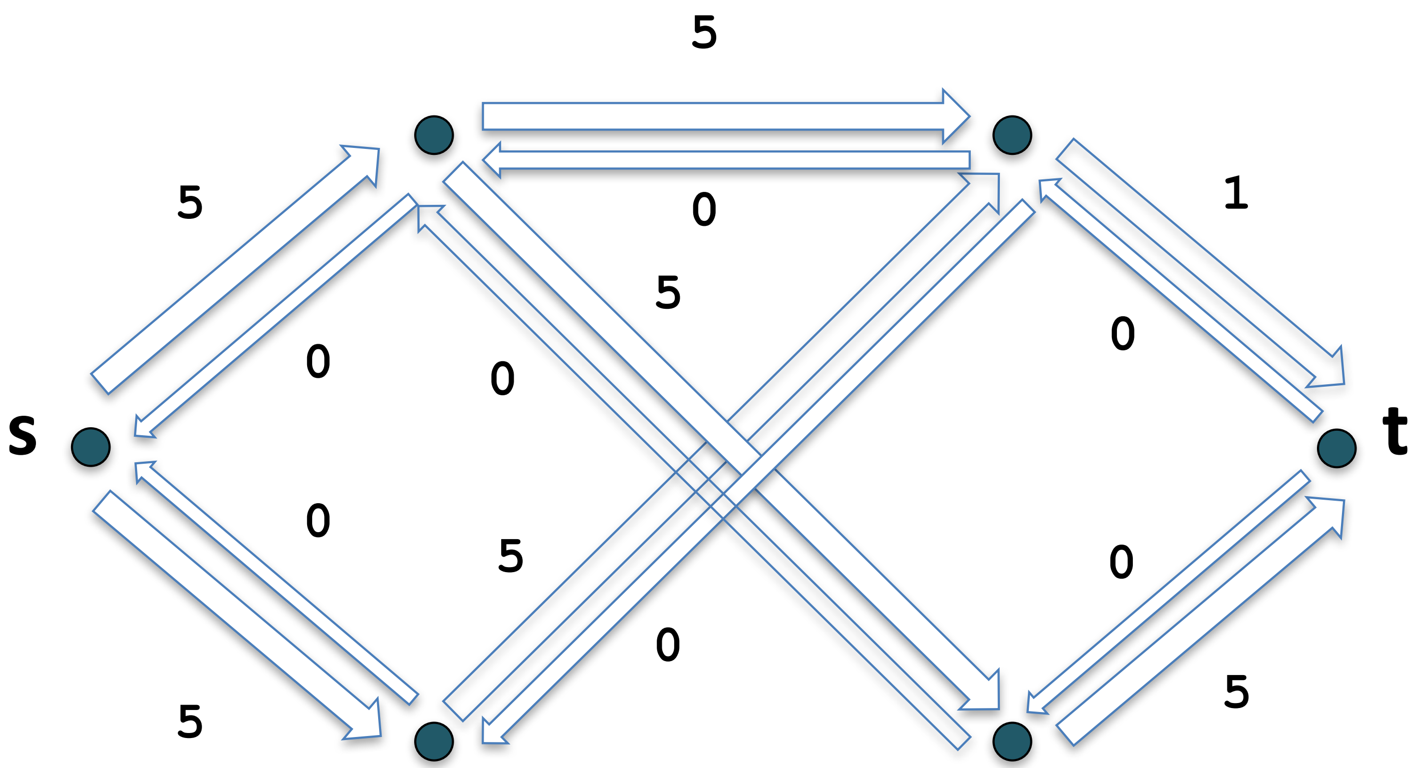
Why for every edge (*u*,*v*): (*v*,*u*)?



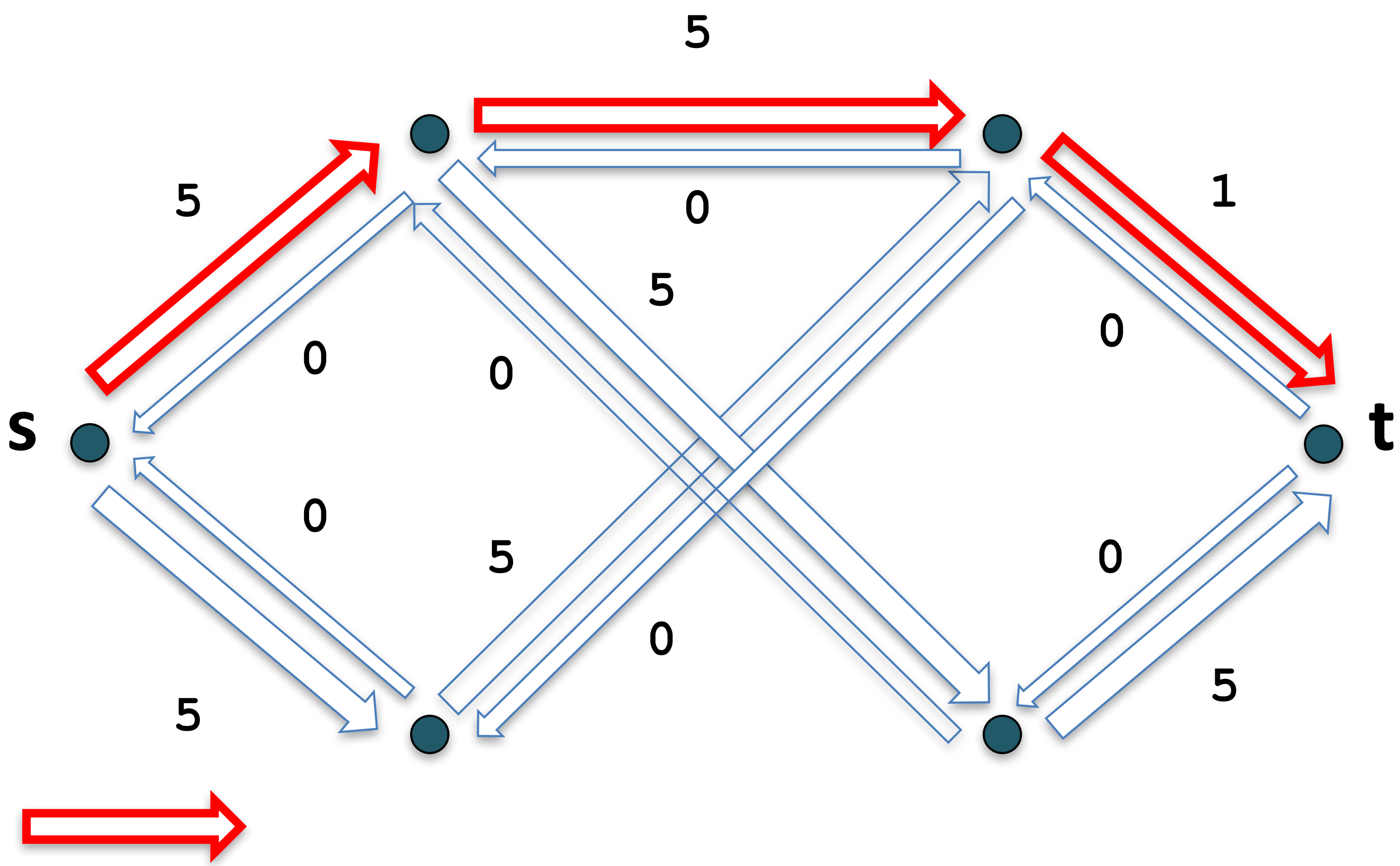
No Augmented Path possible anymore.

OPTIMAL FLOW = 5 ????

For every edge (u,v) an additional (back)edge (v,u) with $c(v,u) = f(u,v)$

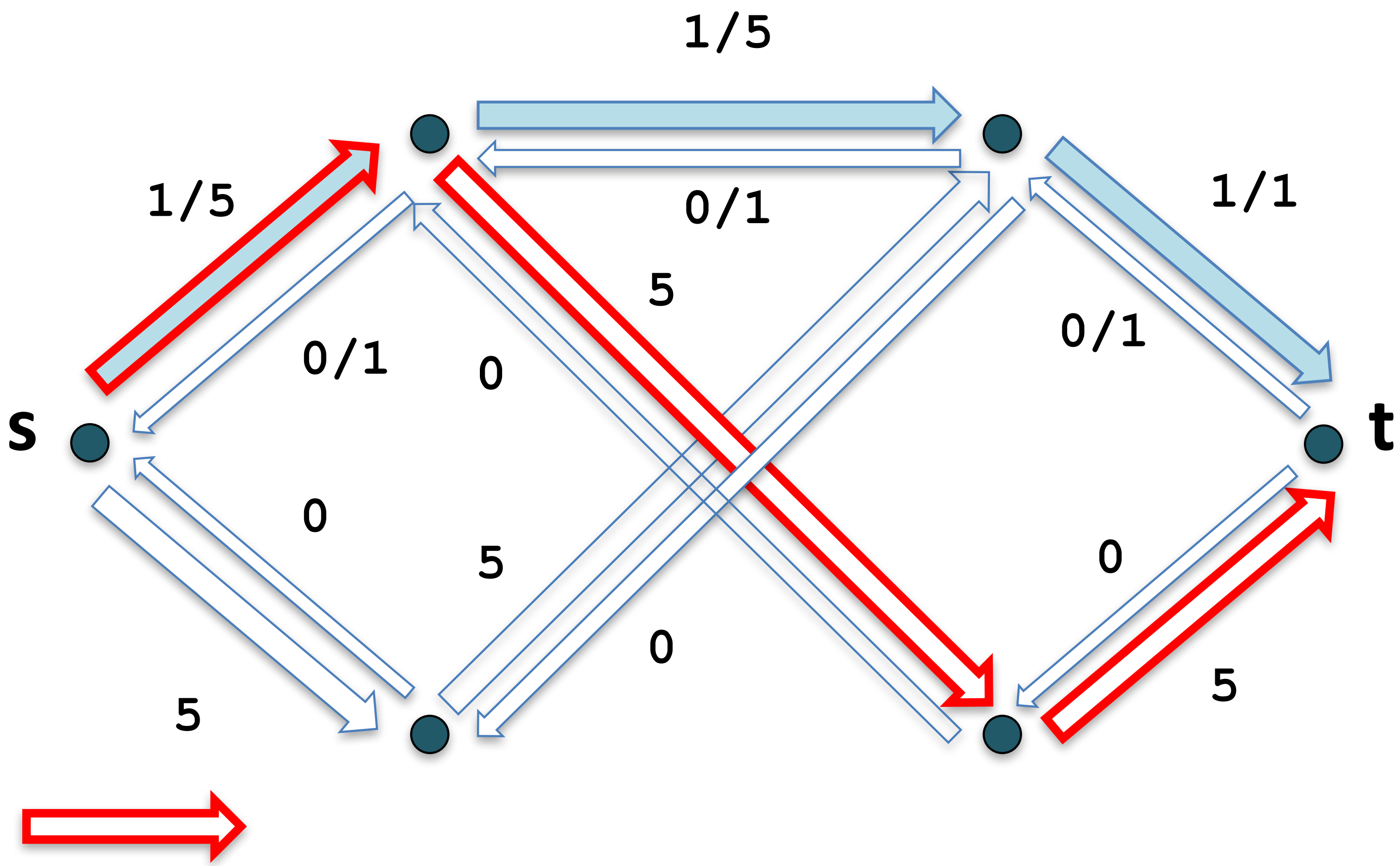


For every edge (u,v) an additional (back)edge (v,u) with $c(v,u) = f(u,v)$



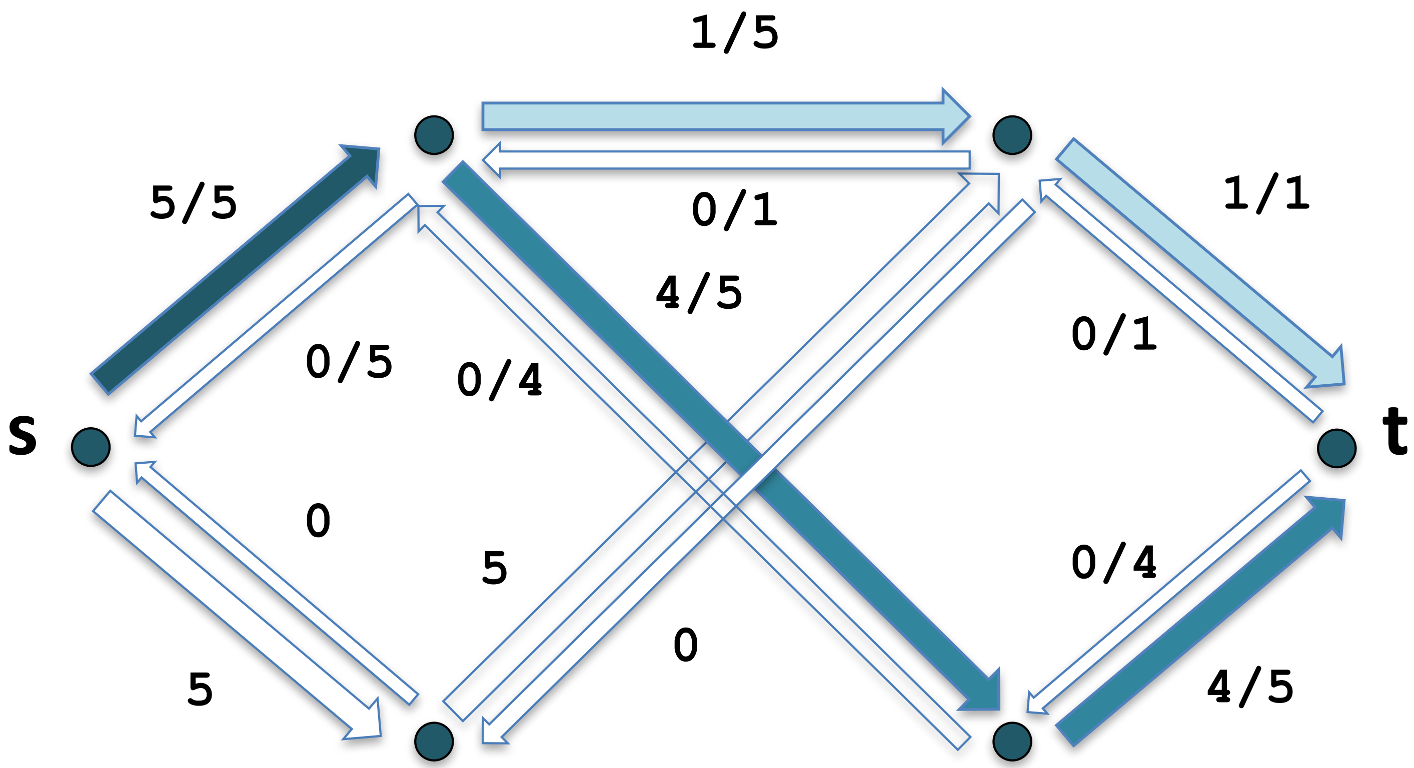
Augmented path with residual capacity = min(5,5,1) = 1

For every edge (u,v) an additional (back)edge (v,u) with $c(v,u) = f(u,v)$



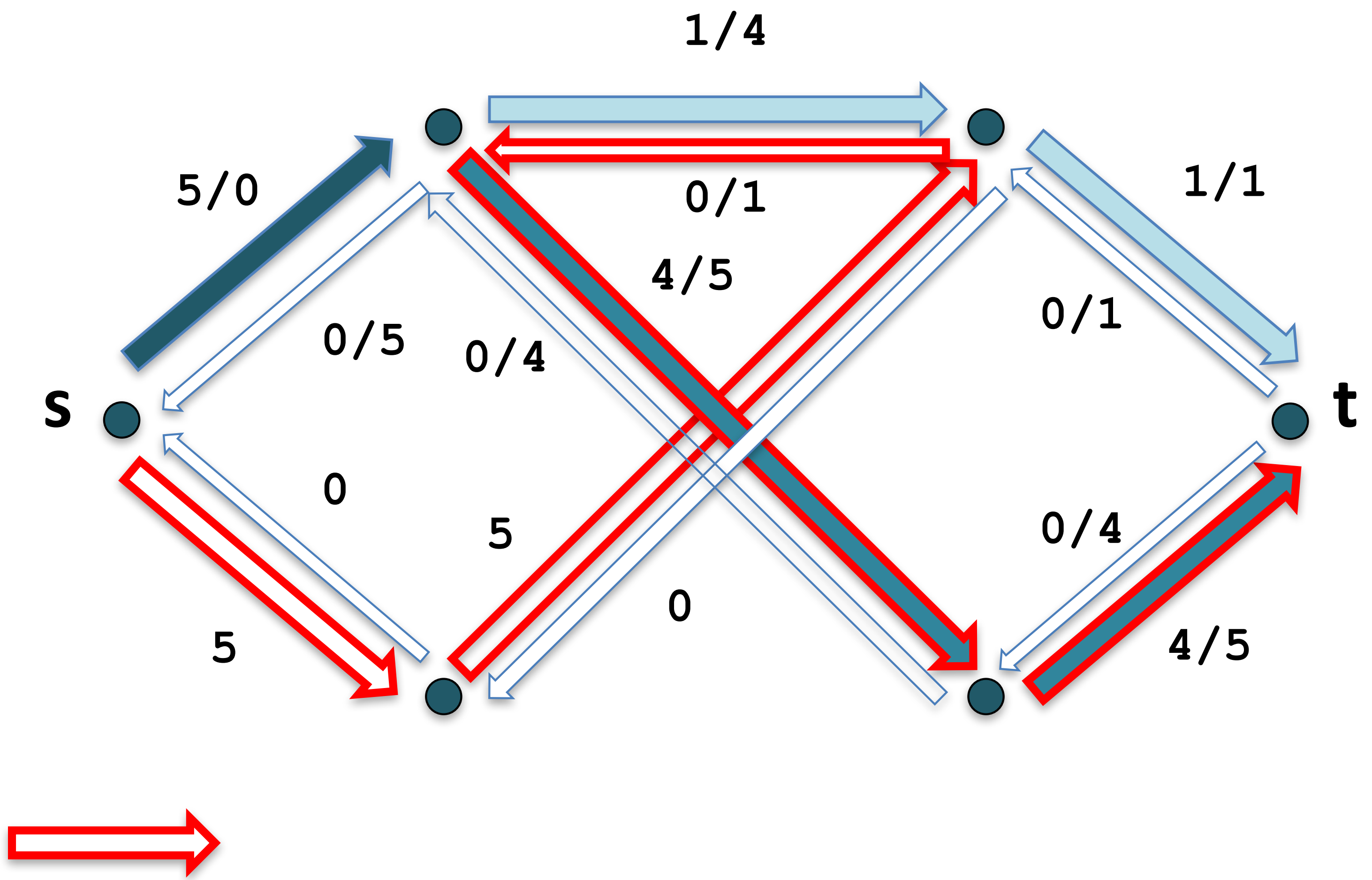
Augmented path with residual capacity = min(4,5,5) = 4

For every edge (u,v) an additional (back)edge (v,u) with $c(v,u) = f(u,v)$



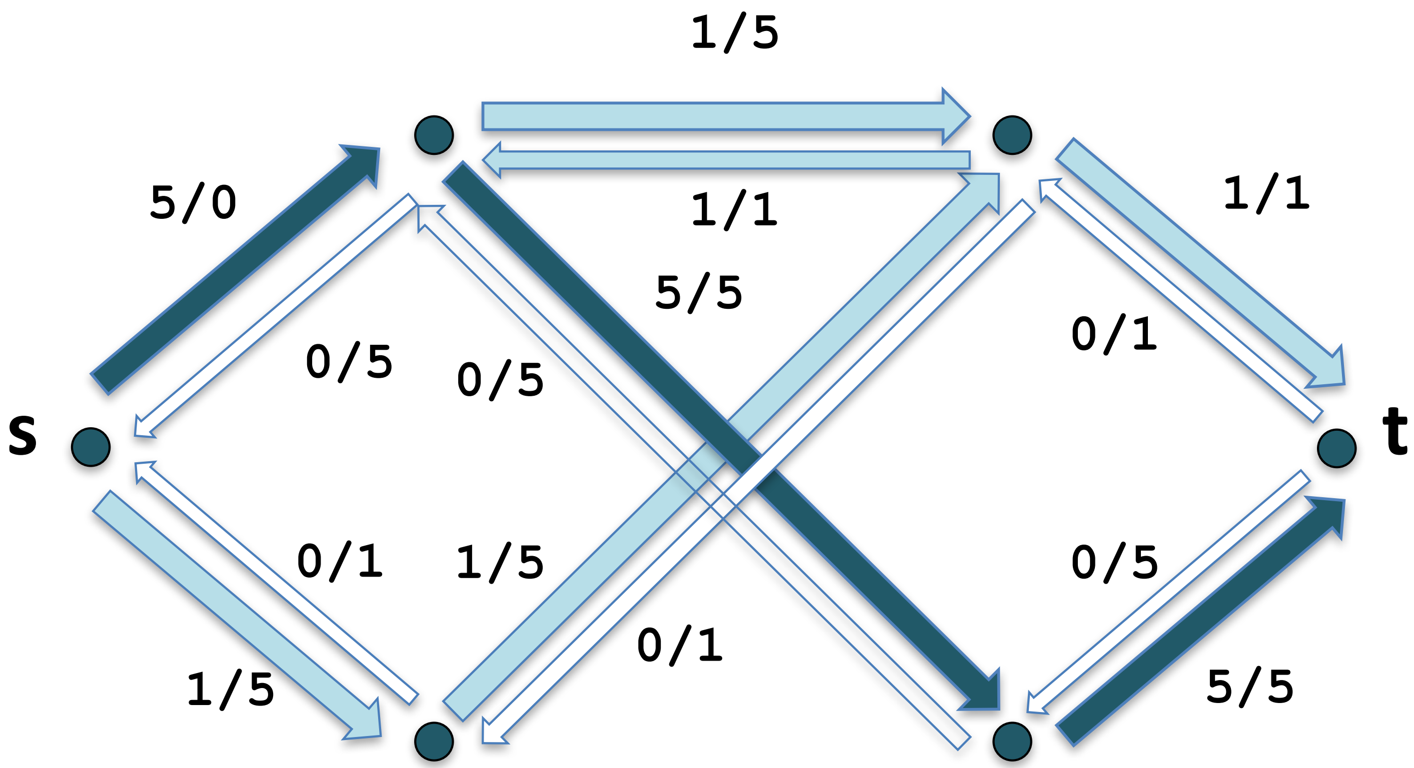
Now Still Augmented Paths POSSIBLE !!!!!!!!

For every edge (u,v) an additional (back)edge (v,u) with $c(v,u) = f(u,v)$



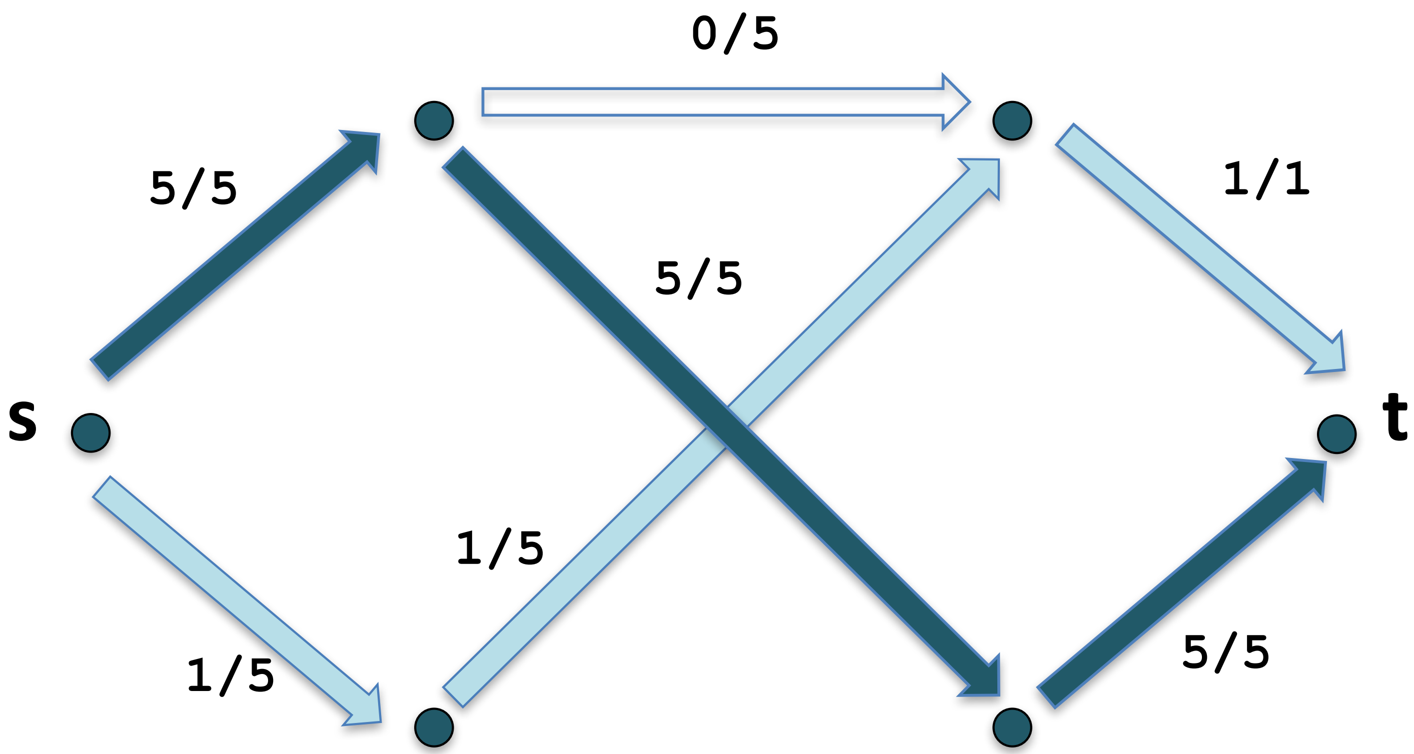
Augmented path with residual capacity = min(5,5,1,1,1) = 1

For every edge (u,v) an additional (back)edge (v,u) with $c(v,u) = f(u,v)$



MaxFlow = 6 !!!!!!

For every edge (u,v) an additional (back)edge (v,u) with $c(v,u) = f(u,v)$

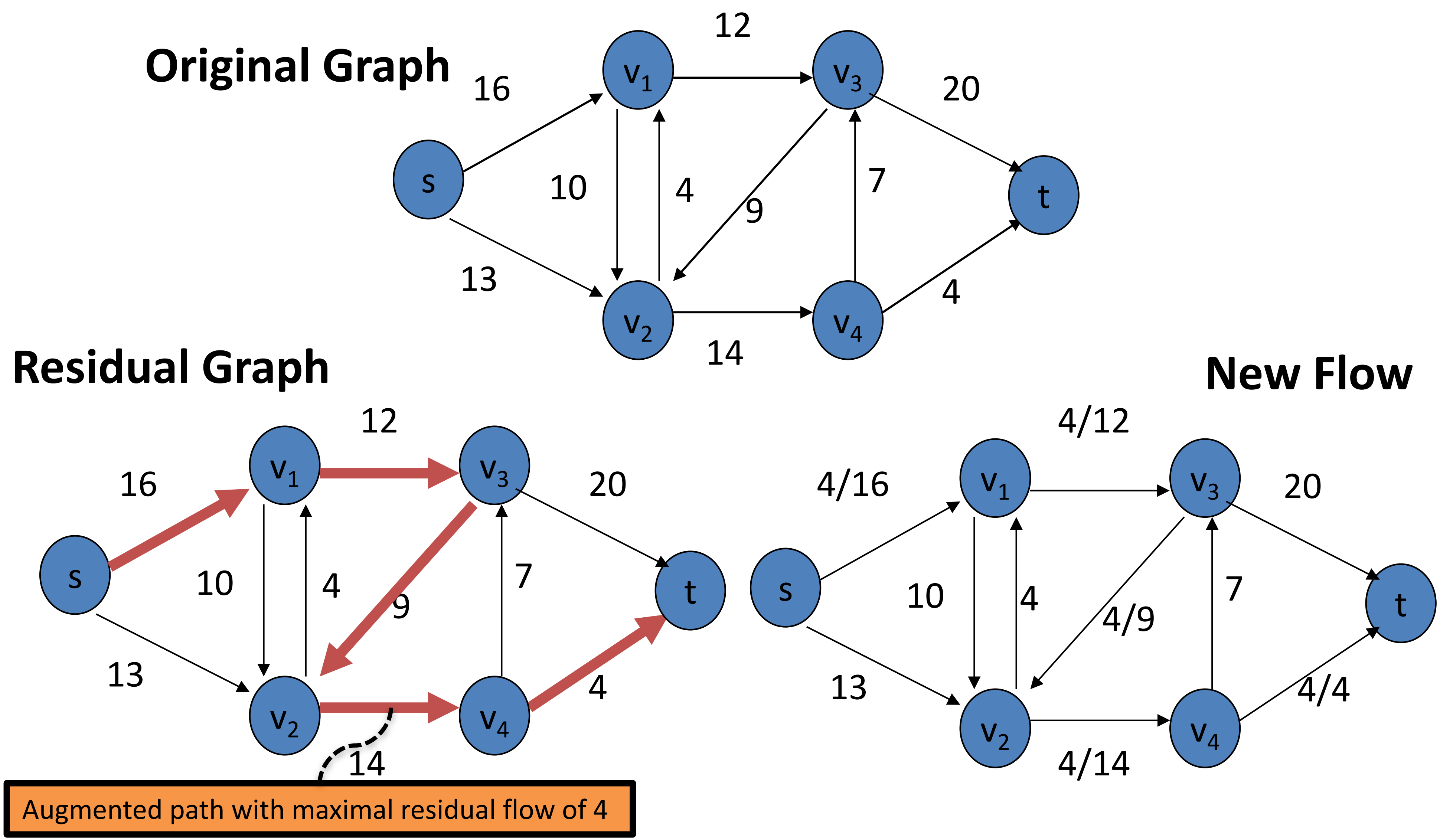


FINAL SOLUTION $f + f'$

More Complex Execution

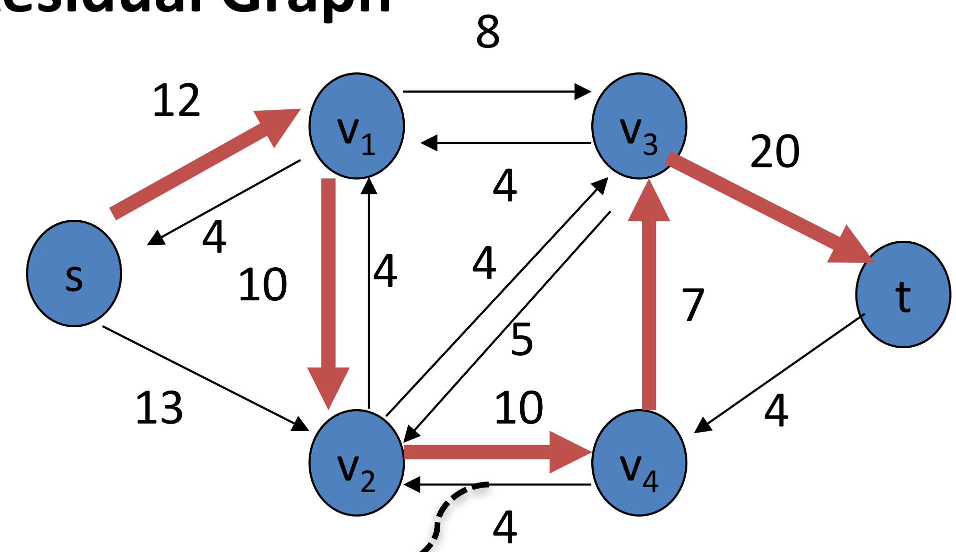
In the following slides:

(a)-(d) Successive iterations of the while loop: The left side of each part shows the residual network G_f from line 4 with a shaded augmenting path p. The right side of each part shows the new flow f that results from adding f_p to f. The residual network in (a) is the input network G. (e) The residual network at the last while loop test. It has no augmenting paths, and the flow f shown in (d) is therefore a maximum flow.


Original Graph
s
v1
v2
v3
v4
t
16
12
20
10
4
9
7
13
14
4
Residual Graph
s
v1
v2
v3
v4
t
16
12
20
10
4
9
7
13
14
4
Augmented path with maximal residual flow of 4
New Flow
s
v1
v2
v3
v4
t
4/16
4/12
20
10
4
4/9
7
13
4/14
4/4

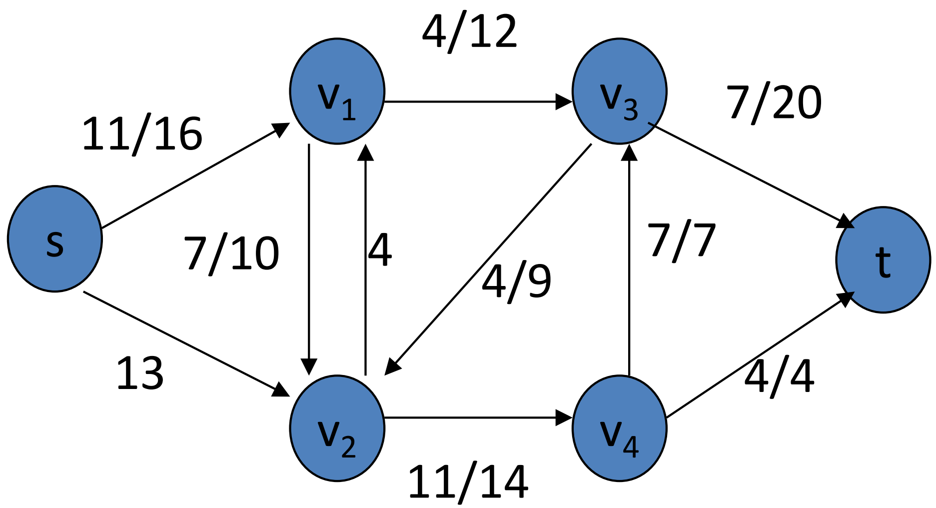
(a)

Residual Graph



Because there is a (forward) flow of 4 on this edge, there is a residual flow capacity of 4 on the back-edge, possibly nullifying the forward flow. The residual of 10 equals the capacity 14 – the forward flow already established 4.

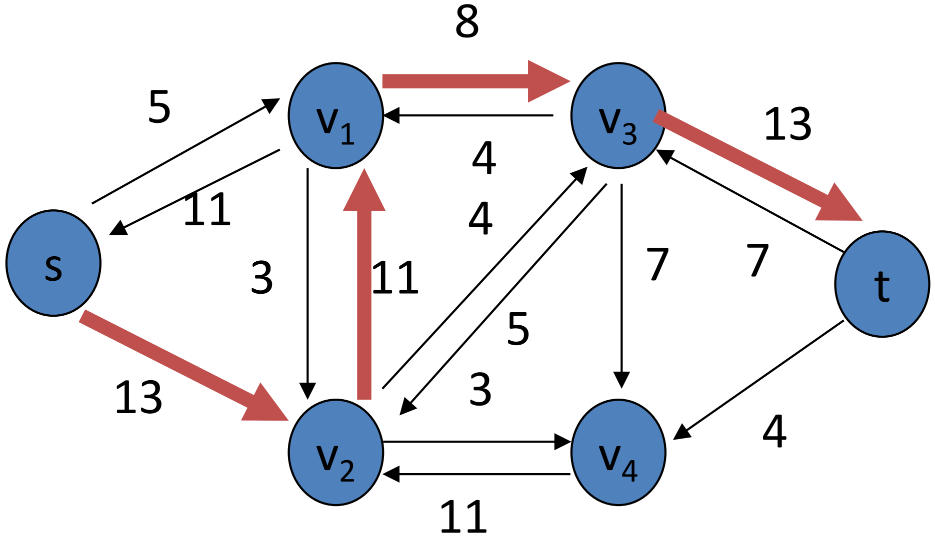
New Flow



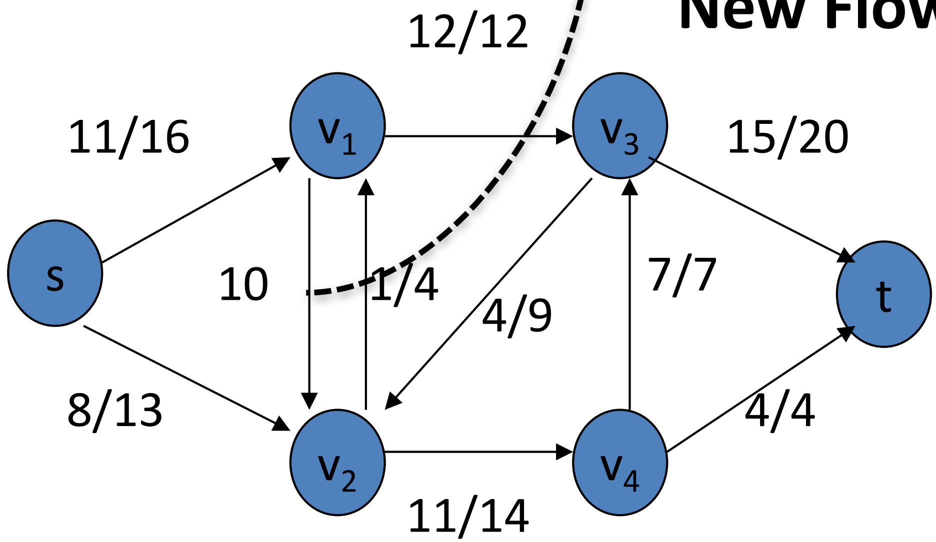
(b)

8 was pushed on the “back edges” from v_1 to v_2 pushing 7 to the edge with capacities 7/10 resulting in (0/)10 and 1 was pushed to the edge with capacities (0/)4 resulting in 1/4.

Residual Graph

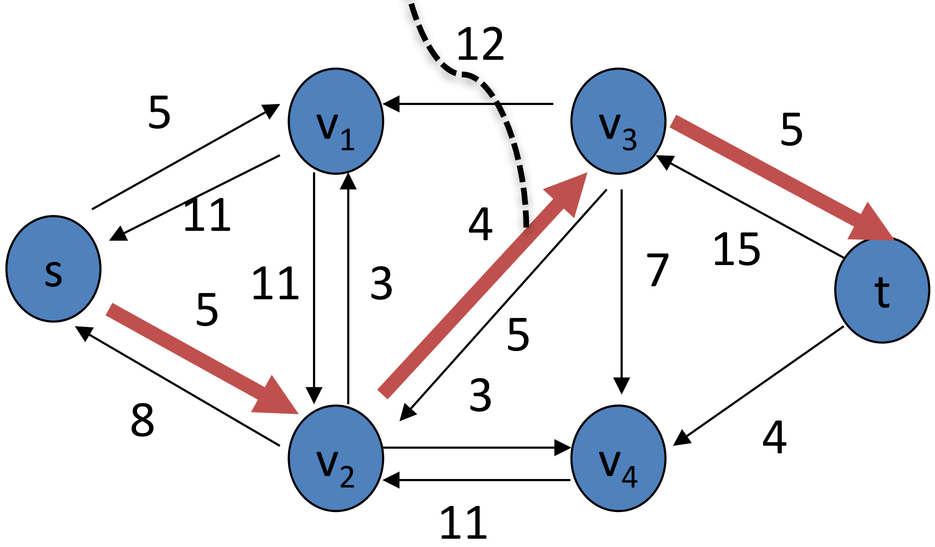


New Flow



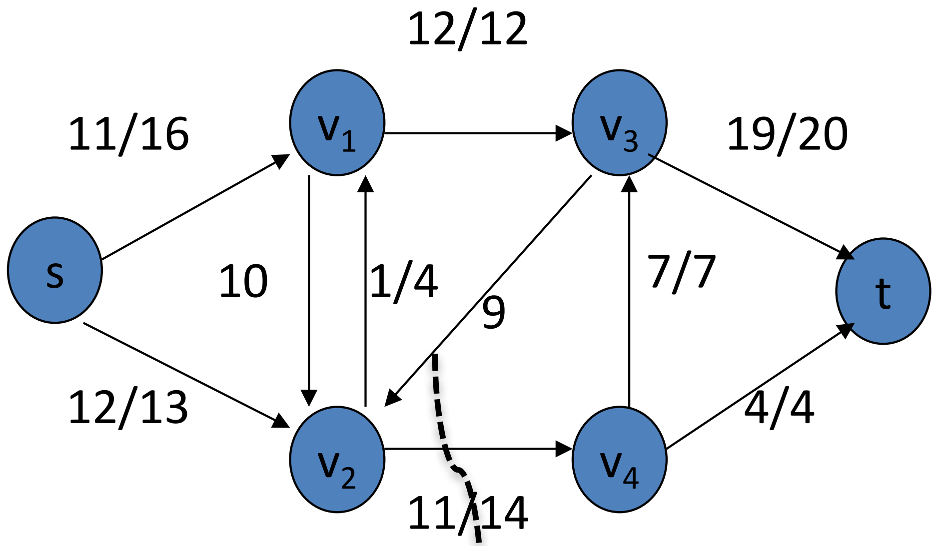
(c)

Original there was no edge (edge with capacity 0) going from v_2 to v_3, but because there was forward flow established on v_3 to v_2, the capacity of (v_2,v_3) was increased to 4!!!!!



Residual Graph

New Flow

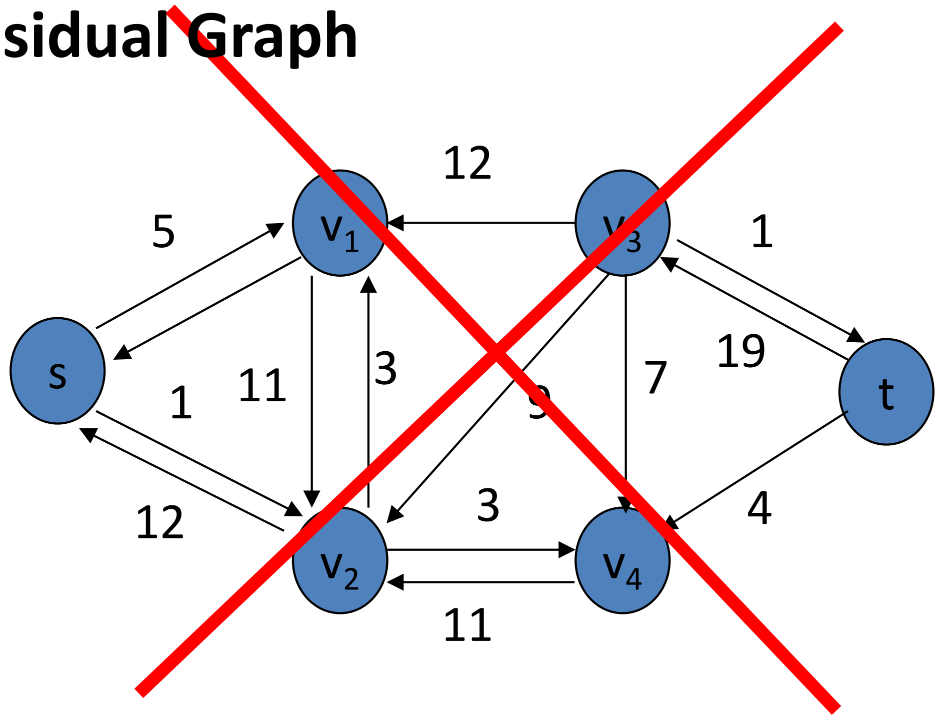


The already established flow of 4 on edge (v_3,v_2) was nullified, thereby increasing the (residual) capacity on this edge to the original value of 9

(d)

Residual Graph

NO AUGMENTED PATH FOUND



(e)

Time Complexity of Ford Fulkerson

$$O(E \max | f |)$$

As long as there is an open path through the residual graph, send the minimum of the residual capacities on the path.

The algorithm is **only guaranteed to terminate if all weights are rational**. Otherwise it is possible that the algorithm will not converge to the maximum value. However, if the algorithm terminates, it is guaranteed to find the maximum value.

The Edmonds-Karp algorithm
A practical implementation of Ford Fulkerson

- Find the augmenting path using breadth-first search.
- Breadth-first search gives the shortest path for graphs. (Assuming the length of each edge is 1.)
- Time complexity of Edmonds-Karp algorithm is $O(VE^2)$.
- The proof is very hard and is not required here.

Relationship with Cut Sets

A cut in a network with source s and sink t is a subset $X \subset V$, such that

$$s \in X \quad \text{and} \quad t \notin X$$

(X, $V \backslash X$) is the set of edges from a vertex in X to a vertex in $V \backslash X$

The capacity of a cut X equals:

$$C(X) = \sum_{x \in (X, V \backslash X)} c(x)$$

➔For every flow f: E -> R and cut X,

$$\boxed{|f| \leq C(X)}$$

Max Flow == Min Cut

Theorem 1: A flow in a network G is maximal iff there exists no augmenting path in G

Theorem 2: The maximal flow in a network G equals the minimal capacity cut set of G

Proof (sketch) Given that f is a maximal flow in G. Construct X such that s ε X, and for all v for which there exists an augmenting path from s to v: v ε X. Then t cannot belong to X, because there is no augmenting path anymore. So X is a proper cut of G. So C (X) = |f| and |f| <= C (Y) for any cut Y. So X is the minimal cut. The reverse follows trivially.

Push-Relabel Algorithm by Goldberg and Tarjan (JACM 1988)

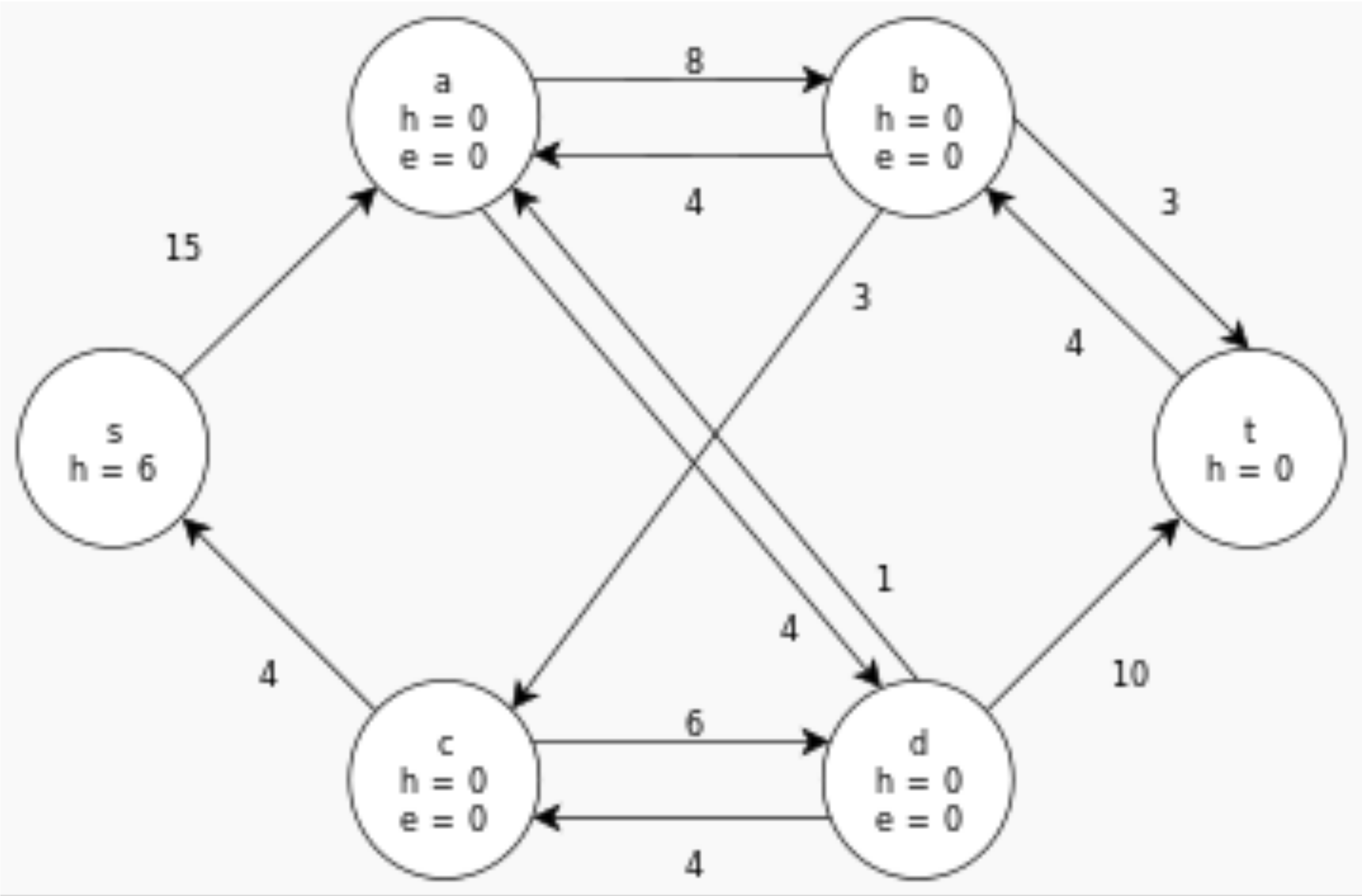
$e_f(v)$ is excess flow in node v

- Input: network $(G = (V, E), s, t, c)$
- $h[s] := |V|$
- for each $v \in V - \{s\}$ do $h[v] := 0$
- for each $(s, v) \in E$ do $f(s, v) := c(s, v)$
- while f is not a feasible flow
  - let $c'(u, v) = c(u, v) - f(u, v) + f(v, u)$ be the capacities of the residual network
  - if there is a vertex $v \in V - \{s, t\}$ and a vertex $w \in V$ such that $e_f(v) > 0$, $h(v) > h(w)$, and $c'(v, w) > 0$ then
    - push $\min\{c'(v, w), e_f(v)\}$ units of flow on the edge (v, w)
  - else, let v be a vertex such that $e_f(v) > 0$, and set $h[v] := h[v] + 1$
- output f

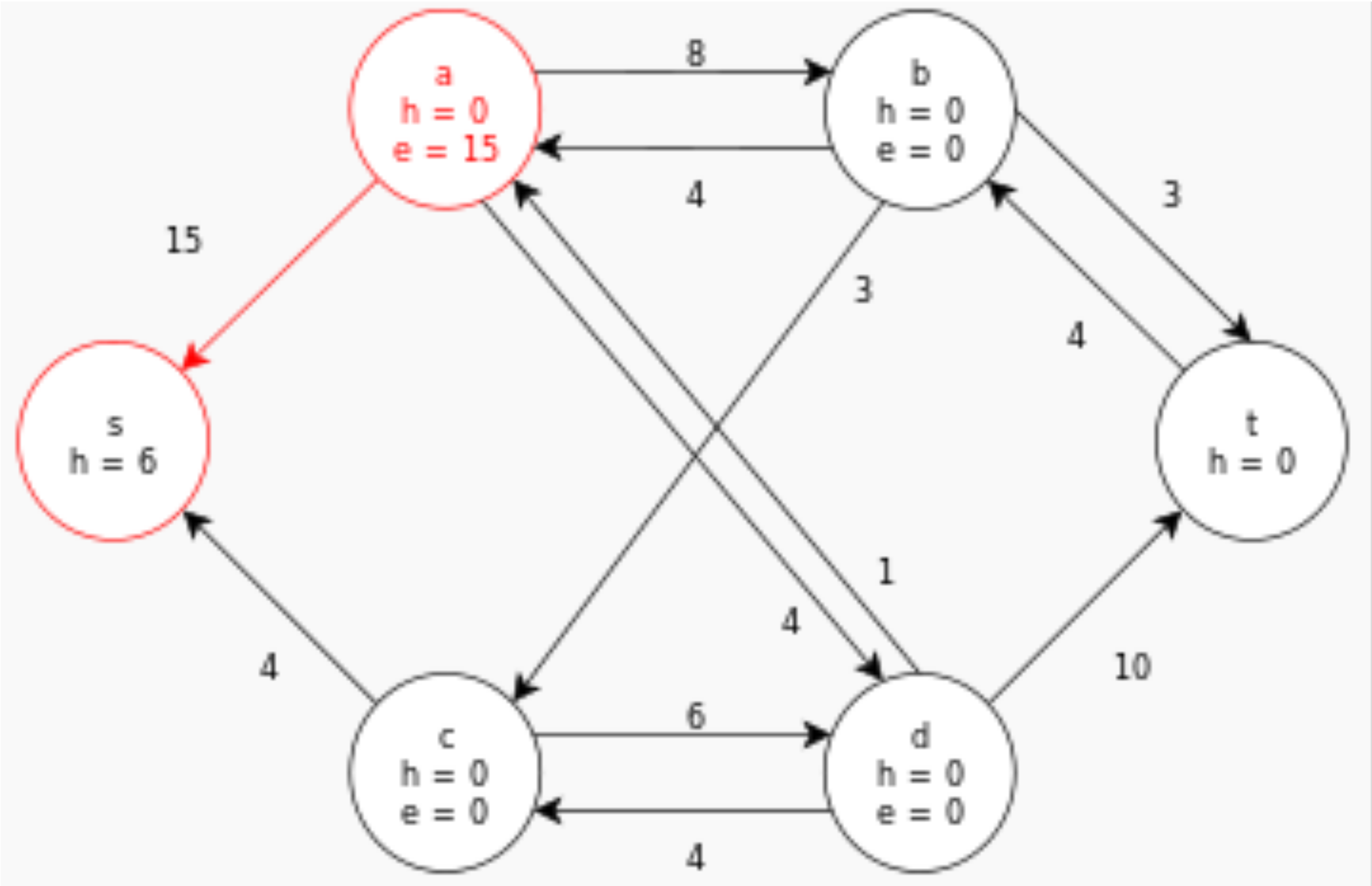
The labeling function h

- Only flow can be pushed from a node v to w if $h(v) > h(w)$
- Once raised, $h(v)$ will never be decremented
- Ping Pong effects are avoided
- The algorithm will actually finish

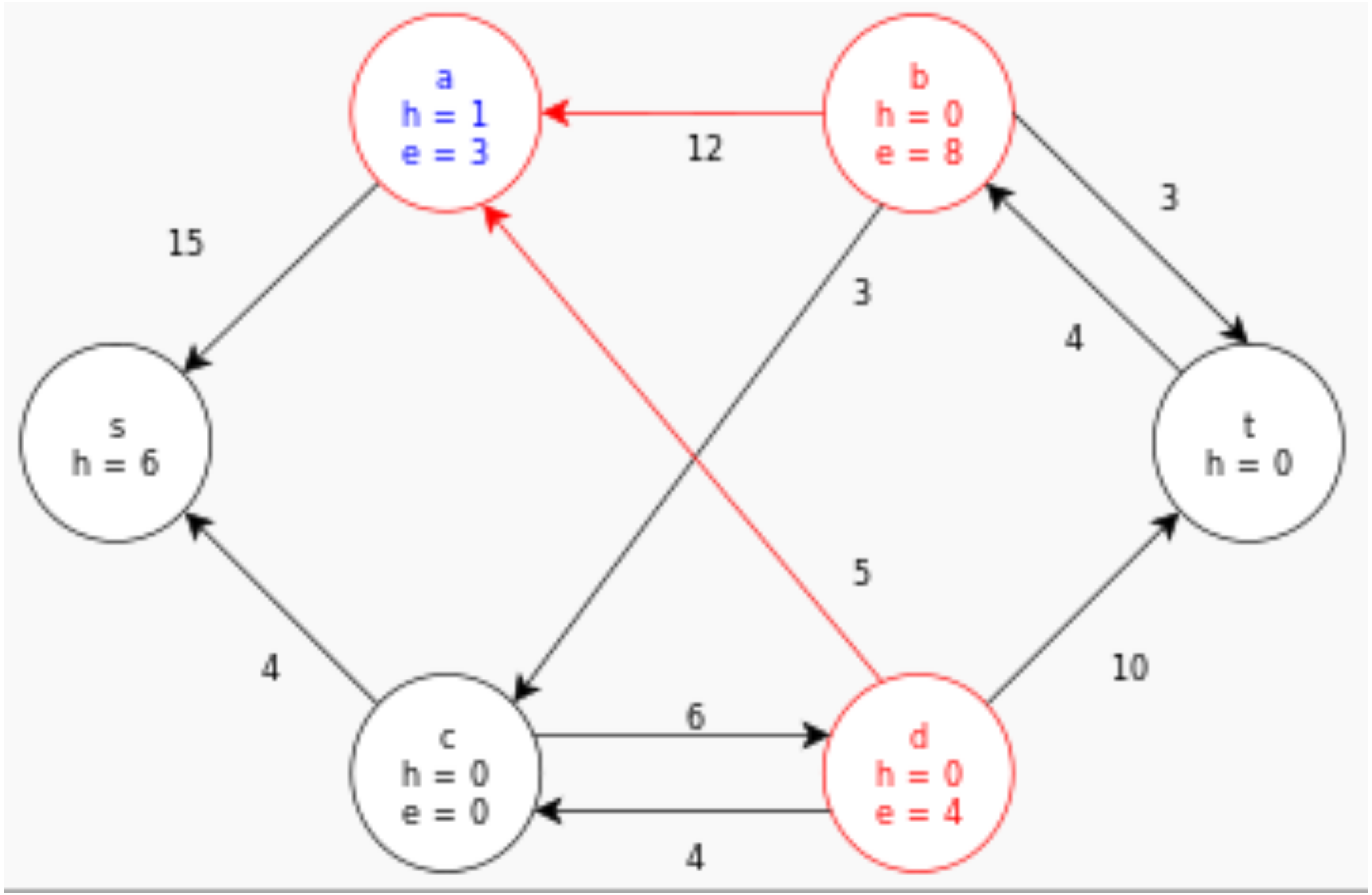
Example



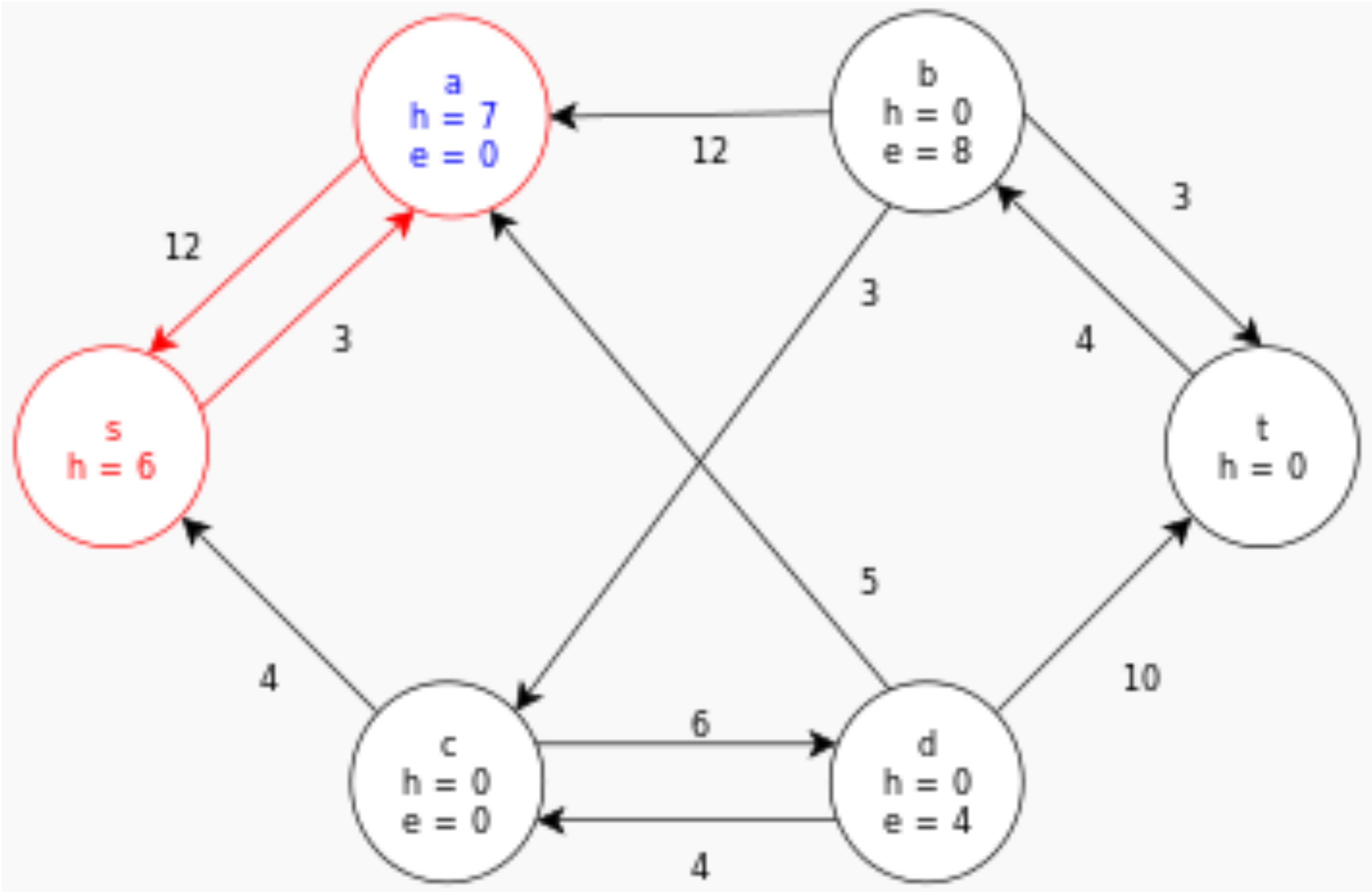
Excess flow is pushed to a



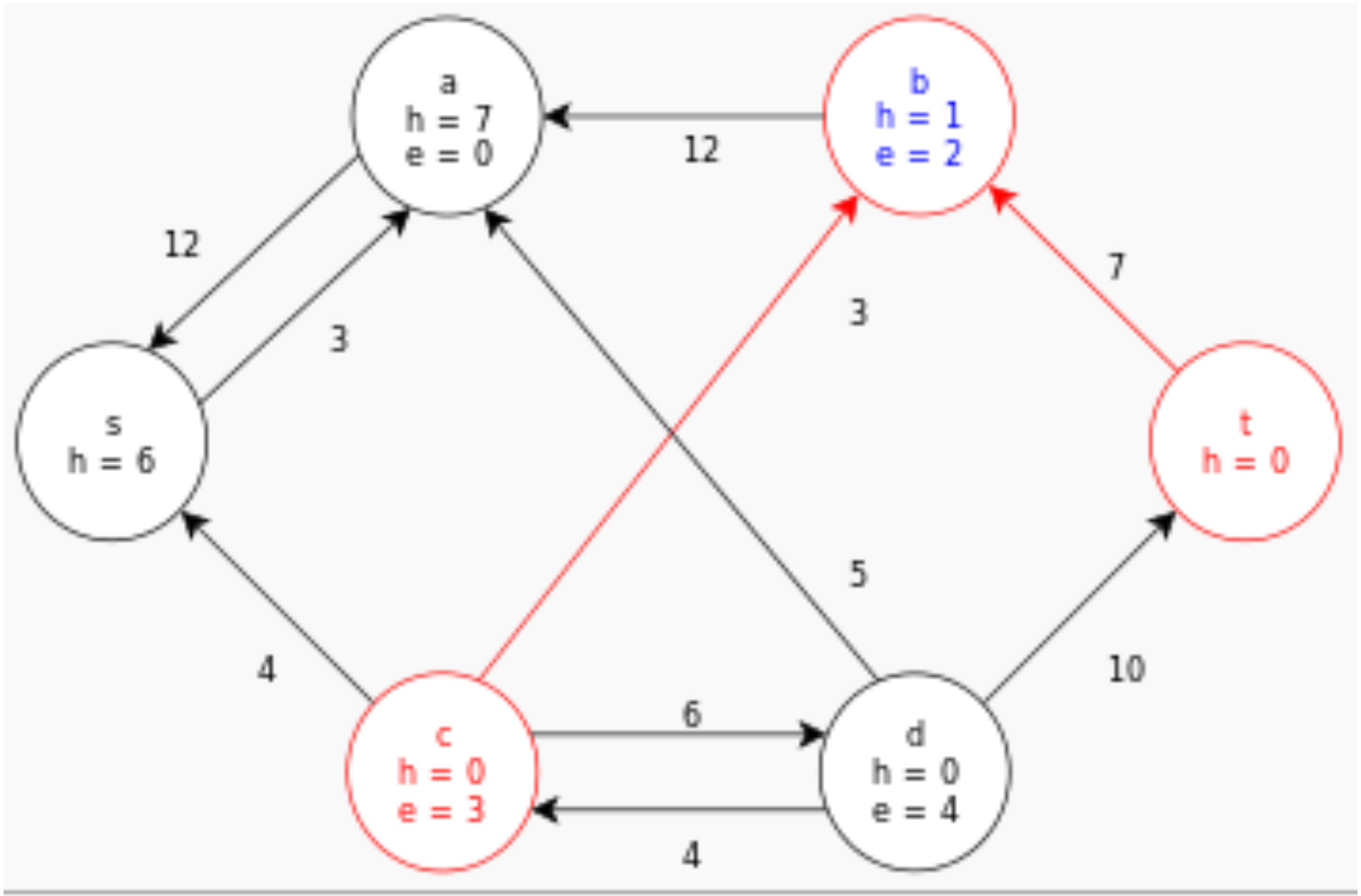
First h[a] is incremented to 1 and then excess flow (12) is pushed from a to b (8) and d (4)



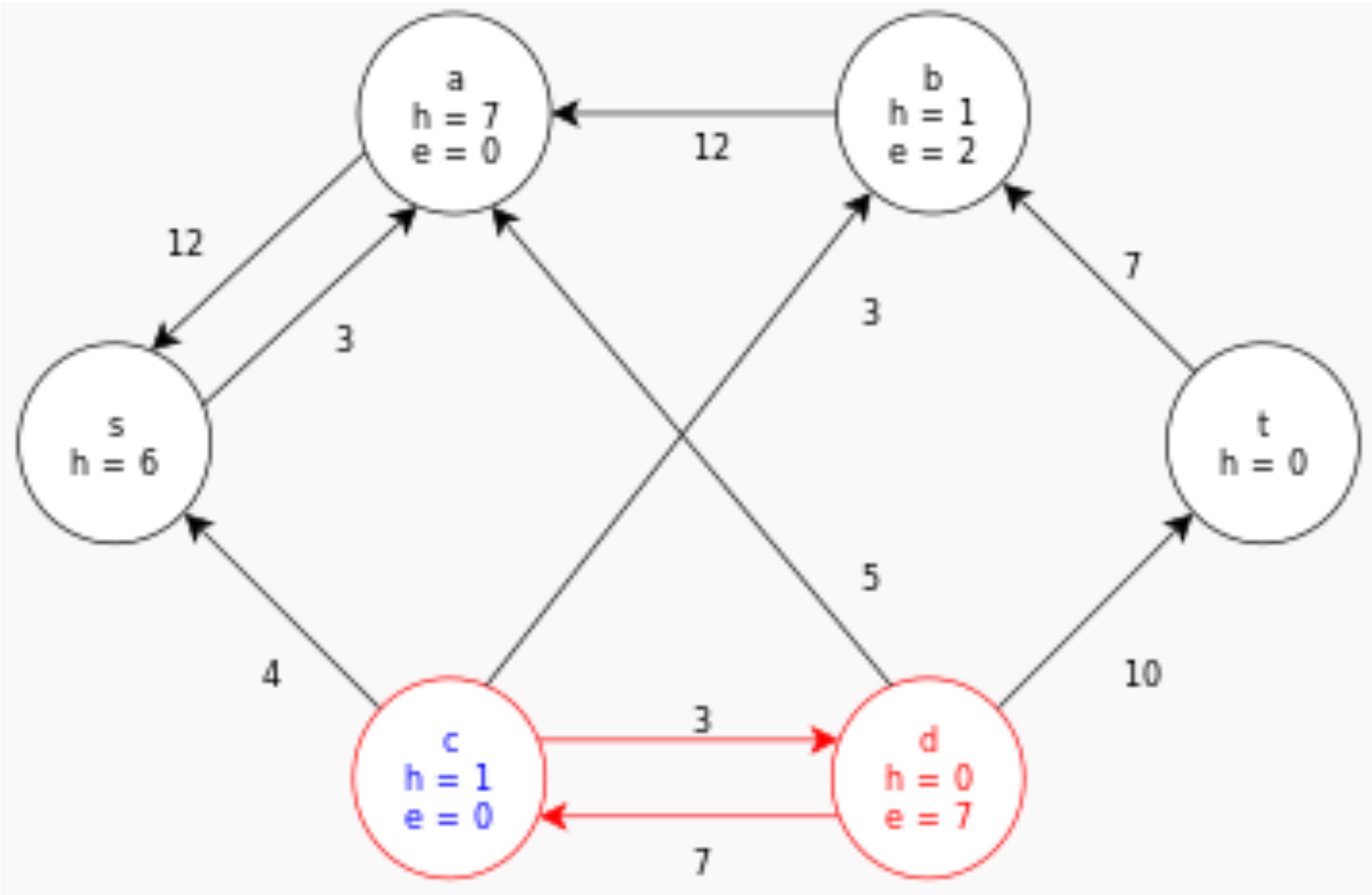
h[a] is incremented to 7!! then excess flow (3) is pushed (back) from a to s



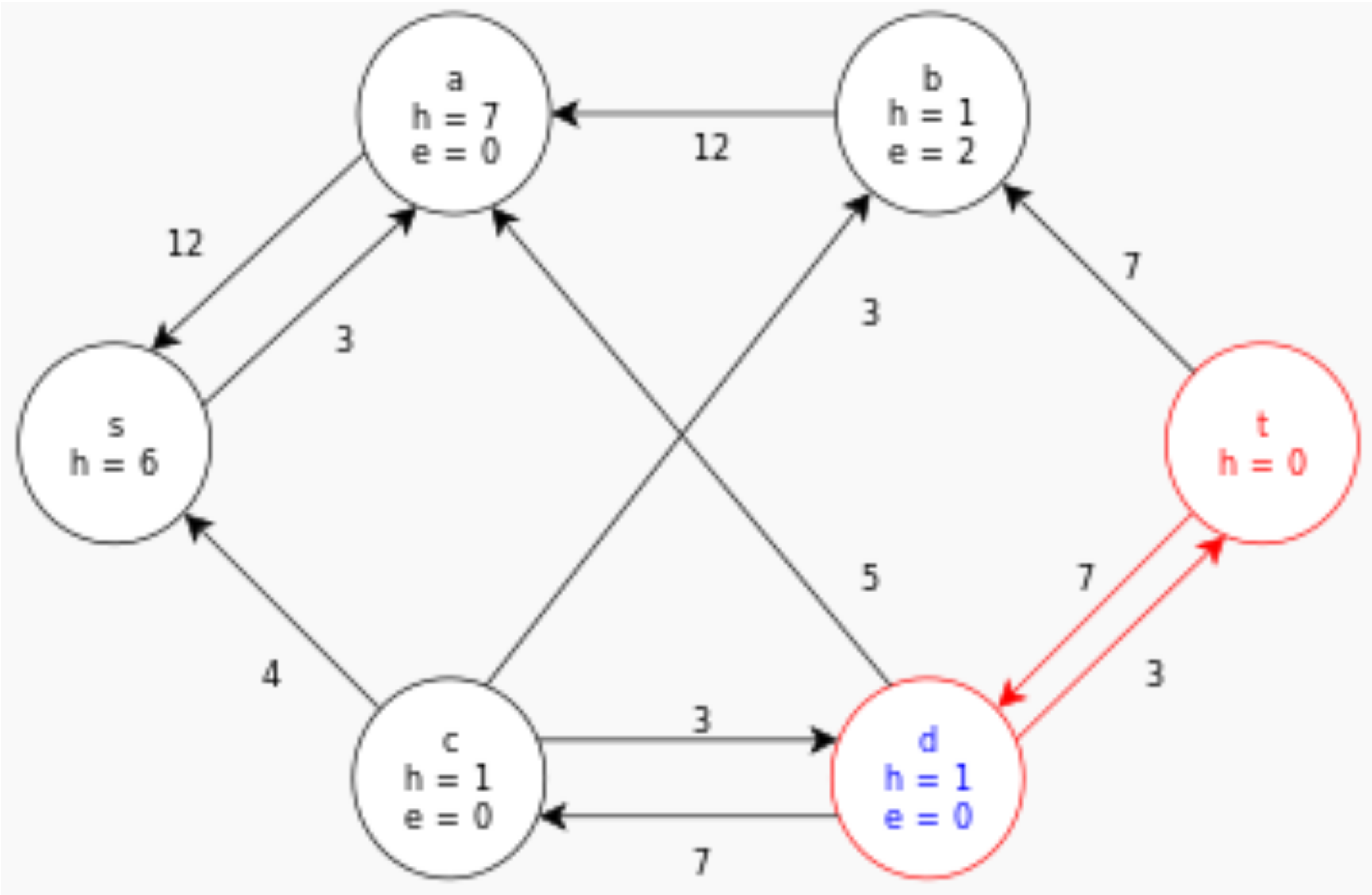
First h[b] is incremented to 1, then excess flow (6) is pushed from b to c (3) and t (3)



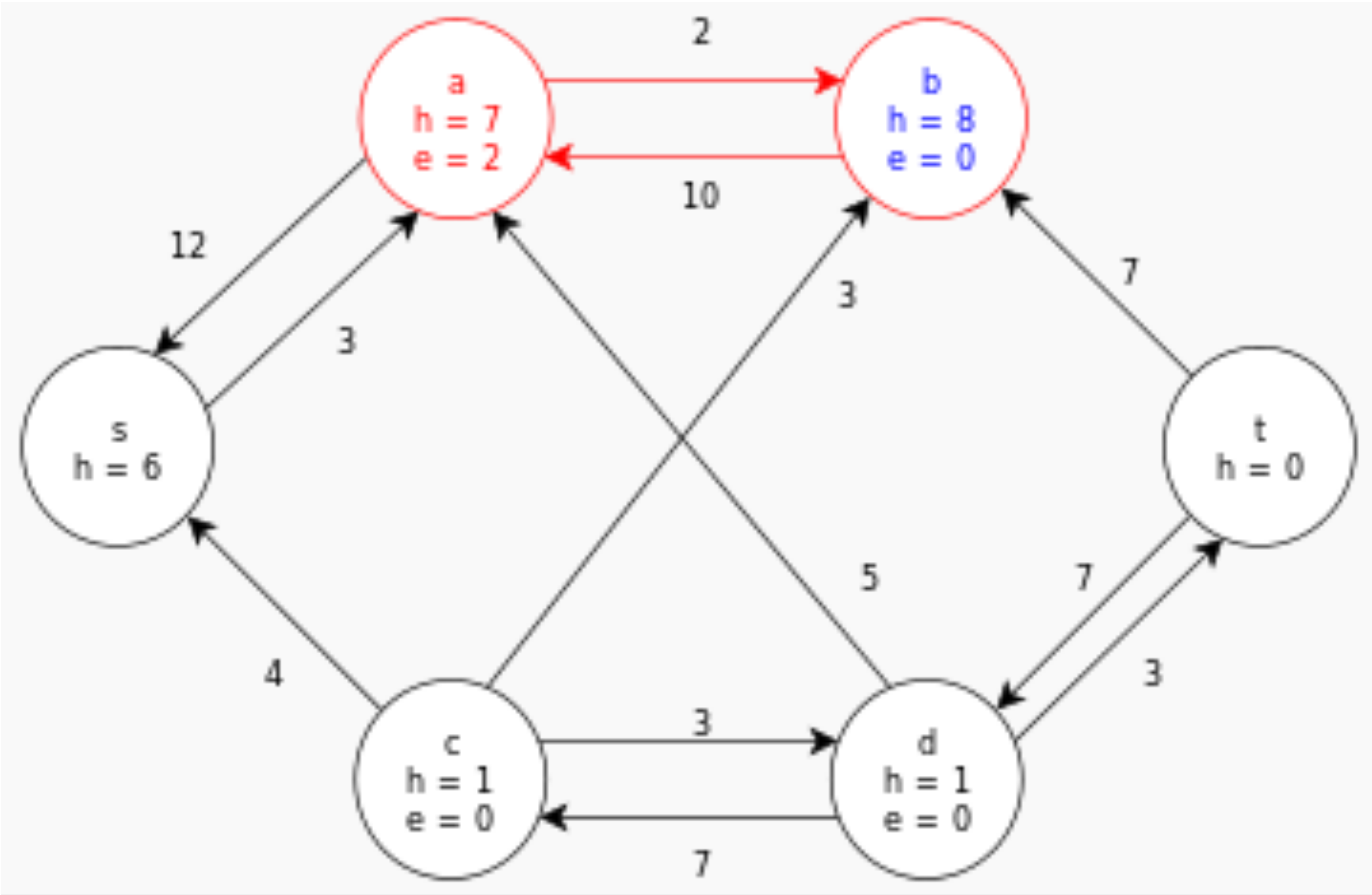
First h[c] is incremented to 1, then excess flow (3) is pushed from c to d



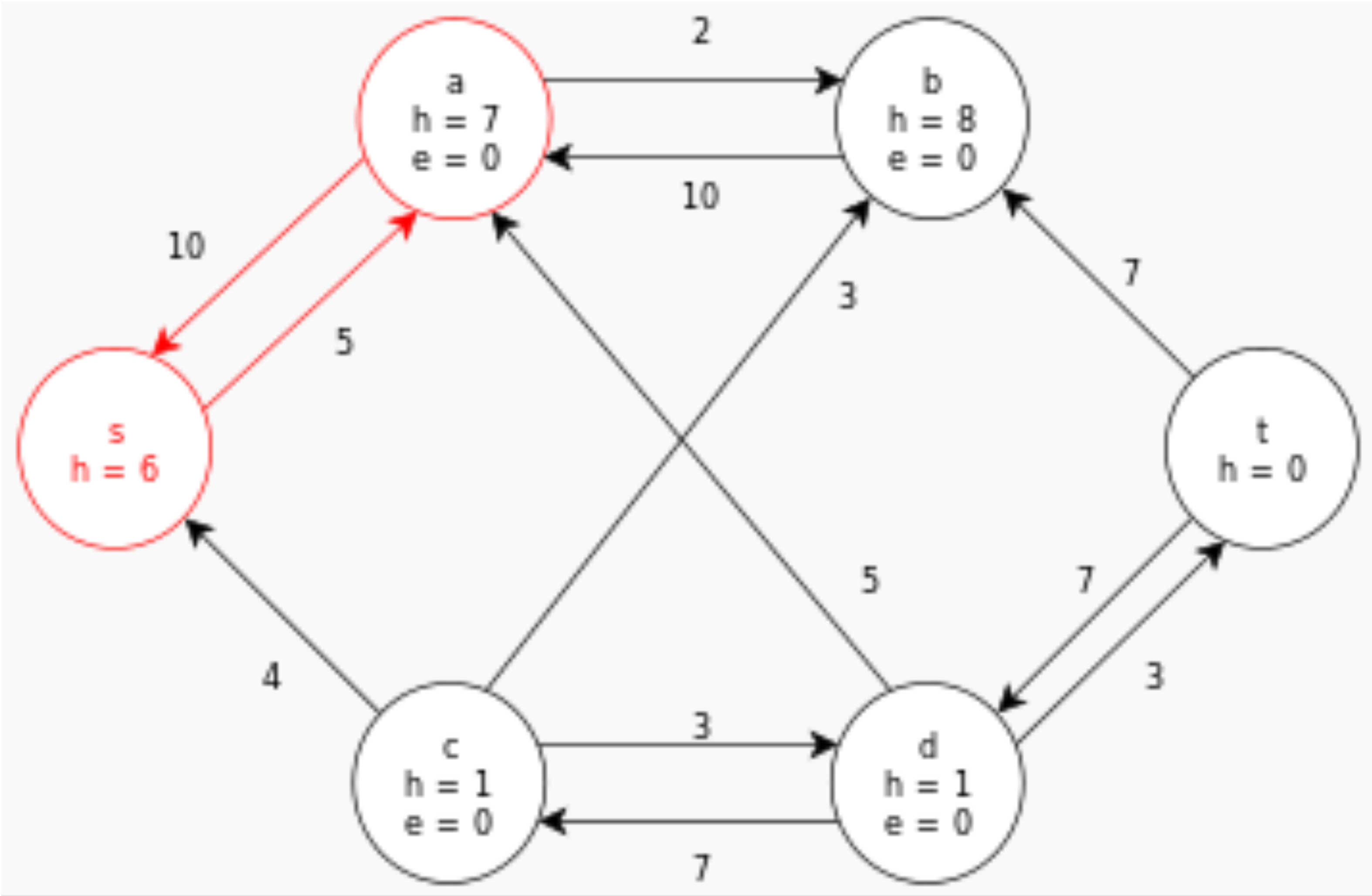
First h[d] is incremented to 1, then excess flow (7) is pushed from d to t



b is the only node with excess > 0, b has no outgoing residual edges, so h[b] is incremented to 8 and b will push **back** excess flow (2) to a



node a is the only active node with excess flow > 0 and will push flow (2)back to s



A parallel version of push relabel