# Theorie van Concurrency

najaar 2011

**http://www.liacs.nl/home/rvvliet/tvc/**

vijftiende college: donderdag 1 december 2011
(laatste hoorcollege)

9. P/T Systems
9.5. Place Invariants
9.6. Marked Graphs
9.7. Free-Choice Systems

zesde en laatste werkcollege: donderdag 8 december 2011

# 9.5. Place Invariants

**Definition 188.** Let $M = (P, T, F, W, C_{in})$ be a P/T system.

A vector $i : P \to \mathbb{Z}$ is a *place invariant* (*p-invariant*) of $M$ if for all configurations $C, D : P \to \mathbb{N}$ and all $t \in T$: if $C[t\rangle D$, then $C \cdot i = D \cdot i$. The *value* of $i$ is the integer $C_{in} \cdot i$.

A p-invariant $i : P \to \mathbb{N}$ is a *positive* p-invariant of $M$.

A p-invariant $i : P \to \{0, 1\}$ is a *characteristic* p-invariant of $M$

A characteristic p-invariant with value 1 is a *sequential component* of $M$.

**Lemma 189.** Let $M = (P, T, F, W, C_{in})$ be a P/T system and let $i : P \to \mathbb{Z}$ be a p-invariant of $M$.

Then $C \cdot i = C_{in} \cdot i$ for all $C \in \mathbb{C}_M$.

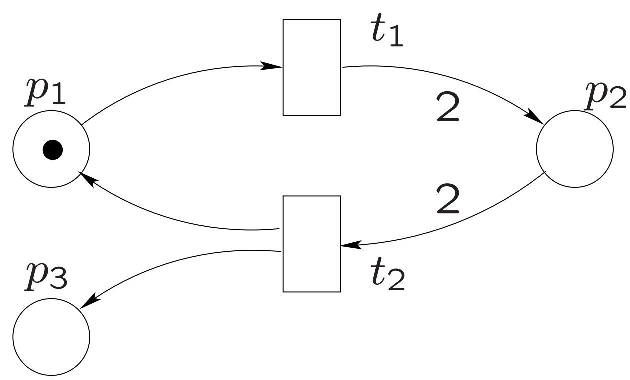



**Fig. 90.** P/T system $M''$ with invariant $2C(p_1) + C(p_2) = 2$.

**Definition 190.** Let $M = (P, T, F, W, C_{in})$ be a P/T system.

(1) For every transition $t \in T$ the vector $\underline{t} : P \to \mathbb{Z}$ is defined as follows:

for every $p \in P$,

| | |
|---|---|
| $\underline{t}(p) = W(t, p)$ | if $p \in t^{\bullet}$, |
| $\underline{t}(p) = -W(p, t)$ | if $p \in {}^{\bullet}t$, and |
| $\underline{t}(p) = 0$ | otherwise. |

(2) The matrix $\underline{M} : T \times P \to \mathbb{Z}$ is defined by
for every $t \in T$ and every $p \in P$, $\underline{M}(t, p) = \underline{t}(p)$.

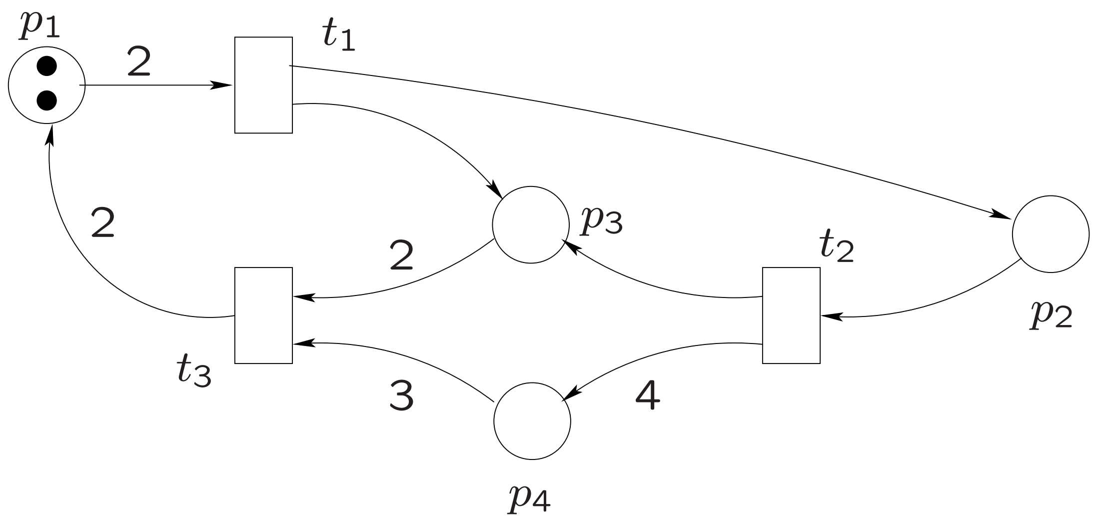



**Fig. 92.** A P/T system $M$ to illustrate linear algebra.

$$\underline{M} = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \\ \begin{pmatrix} -2 & +1 & +1 & 0 \\ 0 & -1 & +1 & +4 \\ +2 & 0 & -2 & -3 \end{pmatrix} \end{array}$$

**Lemma 191.** Let $M$ be a P/T system, let $C$ and $D$ be two configurations of $M$, and let $t \in T_M$.
Then $C[t\rangle D$ iff $D = C + \underline{t}$.

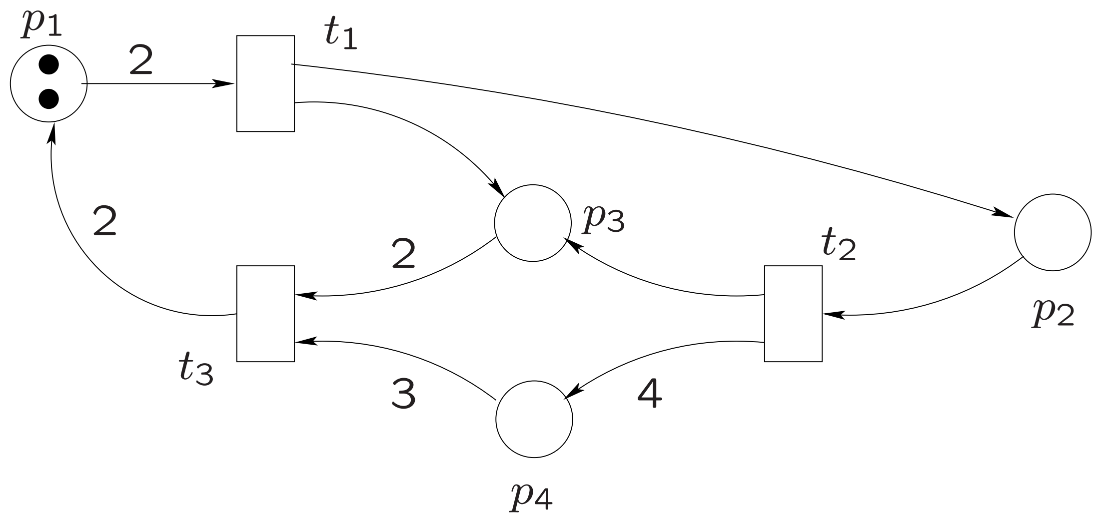



**Fig. 92.** A P/T system $M$ to illustrate linear algebra.

| | $C_{\text{in}}$ | $[t_1\rangle$ | $C_1$ | $[t_2\rangle$ | $C_2$ | $[t_3\rangle$ | $C_3$ | $[t_1\rangle$ | $C_4$ | $[t_2\rangle$ | $C_5$ | $[t_3\rangle$ | $C_6$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | (2) | $-2$ | (0) | 0 | (0) | $+2$ | (2) | $-2$ | (0) | 0 | (0) | $+2$ | (2) |
| $p_2$ | (0) | $+1$ | (1) | $-1$ | (0) | 0 | (0) | $+1$ | (1) | $-1$ | (0) | 0 | (0) |
| $p_3$ | (0) | $+1$ | (1) | $+1$ | (2) | $-2$ | (0) | $+1$ | (1) | $+1$ | (2) | $-2$ | (0) |
| $p_4$ | (0) | 0 | (0) | $+4$ | (4) | $-3$ | (1) | 0 | (1) | $+4$ | (5) | $-3$ | (2) |

$$\underline{M} = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \\ \left( \begin{array}{rrrr} -2 & +1 & +1 & 0 \\ 0 & -1 & +1 & +4 \\ +2 & 0 & -2 & -3 \end{array} \right) \end{array}$$

| | $C_{\text{in}}$ | $[t_1\rangle$ | $C_1$ | $[t_2\rangle$ | $C_2$ | $[t_3\rangle$ | $C_3$ | $[t_1\rangle$ | $C_4$ | $[t_2\rangle$ | $C_5$ | $[t_3\rangle$ | $C_6$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | (2) | −2 | (0) | 0 | (0) | +2 | (2) | −2 | (0) | 0 | (0) | +2 | (2) |
| $p_2$ | (0) | +1 | (1) | −1 | (0) | 0 | (0) | +1 | (1) | −1 | (0) | 0 | (0) |
| $p_3$ | (0) | +1 | (1) | +1 | (2) | −2 | (0) | +1 | (1) | +1 | (2) | −2 | (0) |
| $p_4$ | (0) | 0 | (0) | +4 | (4) | −3 | (1) | 0 | (1) | +4 | (5) | −3 | (2) |

**Theorem 192.** Let $M = (P, T, F, W, C_{in})$ be a P/T system and let $i : P \to \mathbb{Z}$. Then the following three statements are equivalent.
(1) $i$ is a p-invariant;
(2) $\underline{t} \cdot i = 0$ for all $t \in T$;
(3) $\underline{M} \cdot i = 0$.
If, moreover, $M$ is reduced, then this is also equivalent with
(4) $C \cdot i = C_{in} \cdot i$ for all $C \in \mathbb{C}_M$.

**Corollary 193.** Let $M = (P, T, F, W, C_{in})$ be a P/T system and let $i : P \to \mathbb{N}$ be the characteristic function of $S \subseteq P$.

Then $i$ is a (characteristic) p-invariant iff

$\sum\{W(p,t) \mid p \in {}^{\bullet}t \cap S\} = \sum\{W(t,p) \mid p \in t^{\bullet} \cap S\}$ for all $t \in T$.

If, in particular, $W(x,y) = 1$ for all $(x,y) \in F$, then this is equivalent with: $\#({}^{\bullet}t \cap S) = \#(t^{\bullet} \cap S)$ for all $t \in T$.

**Theorem 49.** Let $M = (P, T, F, C_{in})$ be a reduced EN system and let $S \subseteq P$.
Then the following statements are equivalent.

(1) There is a sequential component $M'$ of $M$ with $P_{M'} = S$,

(2) $\#(C \cap S) = 1$ for all $C \in \mathbb{C}_M$,

(3) (i) $\#(C_{in} \cap S) = 1$, and
(ii) $\forall t \in T$ :
$\#({}^\bullet t \cap S) = \#(t^\bullet \cap S) = 1$ or $\#({}^\bullet t \cap S) = \#(t^\bullet \cap S) = 0$.

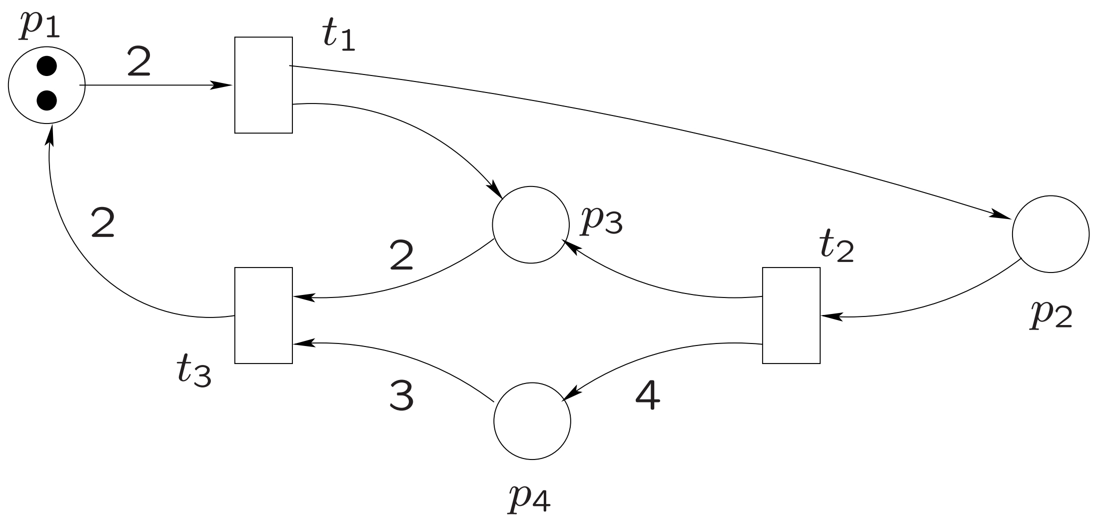



**Fig. 92.** A P/T system $M$ to illustrate linear algebra.

$$\underline{M} = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \\ \begin{pmatrix} -2 & +1 & +1 & 0 \\ 0 & -1 & +1 & +4 \\ +2 & 0 & -2 & -3 \end{pmatrix} \end{array}$$

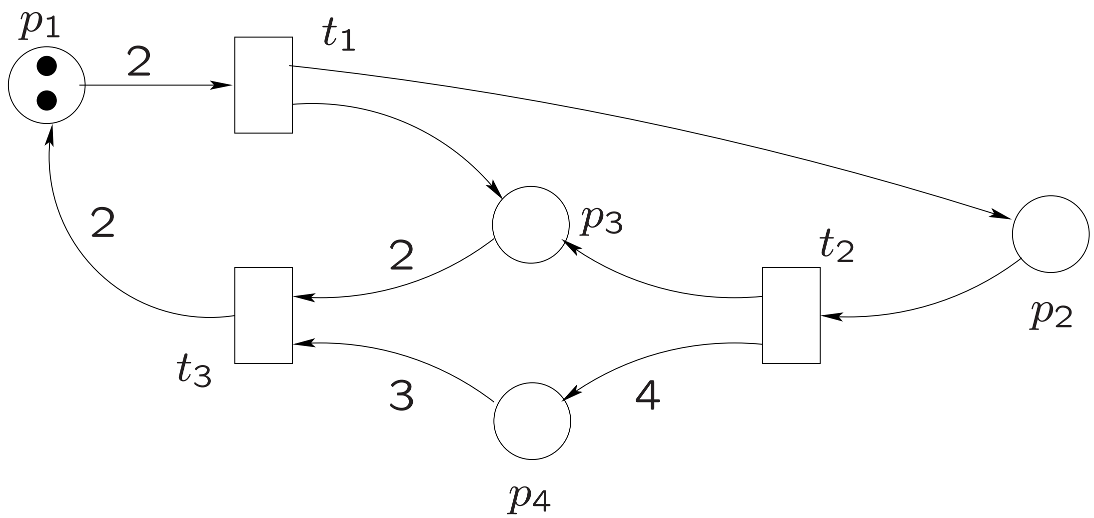



**Fig. 92.** A P/T system $M$ to illustrate linear algebra.

$$\underline{M \cdot i} = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \\ \begin{pmatrix} -2 & +1 & +1 & 0 \\ 0 & -1 & +1 & +4 \\ +2 & 0 & -2 & -3 \end{pmatrix} \end{array} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

**Lemma 194.** Let $M$ be a P/T system and let $\lambda \in \mathbb{Z}$.

If $i_1$ and $i_2$ are p-invariants of $M$, then so are $i_1 + i_2$ and $\lambda \cdot i_1$.

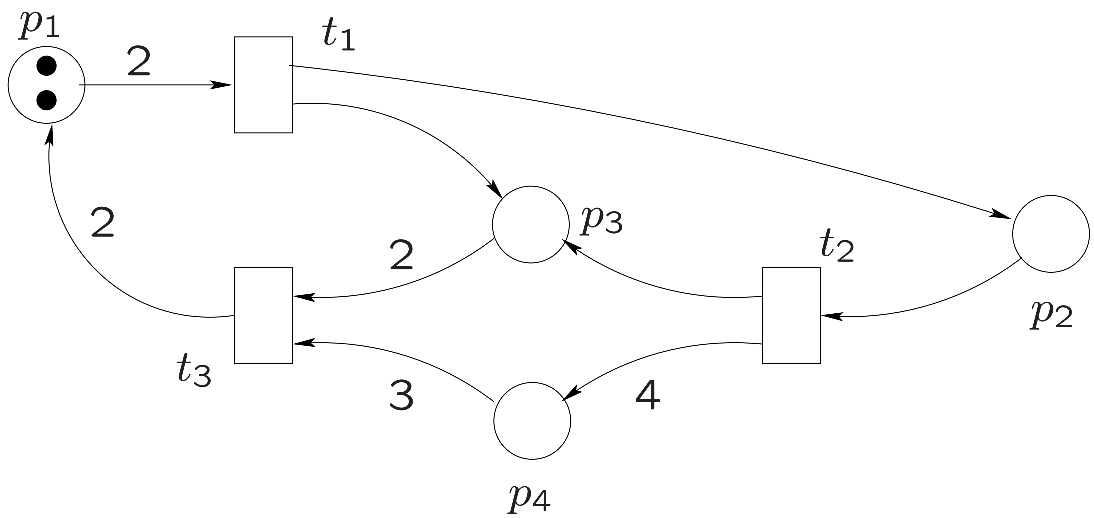



**Fig. 92.** A P/T system $M$ to illustrate linear algebra.

$$\underline{M \cdot i} = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \\ \begin{pmatrix} -2 & +1 & +1 & 0 \\ 0 & -1 & +1 & +4 \\ +2 & 0 & -2 & -3 \end{pmatrix} \end{array} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{M \cdot i} = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \\ \begin{pmatrix} -2 & +1 & +1 & 0 \\ 0 & -1 & +1 & +4 \\ +2 & 0 & -2 & -3 \end{pmatrix} \end{array} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In other words:

$$\begin{array}{rcrcrcrcl} -2x_1 & + & x_2 & + & x_3 & & & = & 0, \\ & - & x_2 & + & x_3 & + & 4x_4 & = & 0, \\ +2x_1 & & & - & 2x_3 & - & 3x_4 & = & 0. \end{array}$$

Only solutions: $(x_1, x_2, x_3, x_4) = \lambda \cdot (1, 1, 1, 0)$ with $\lambda \in \mathbb{Z}$

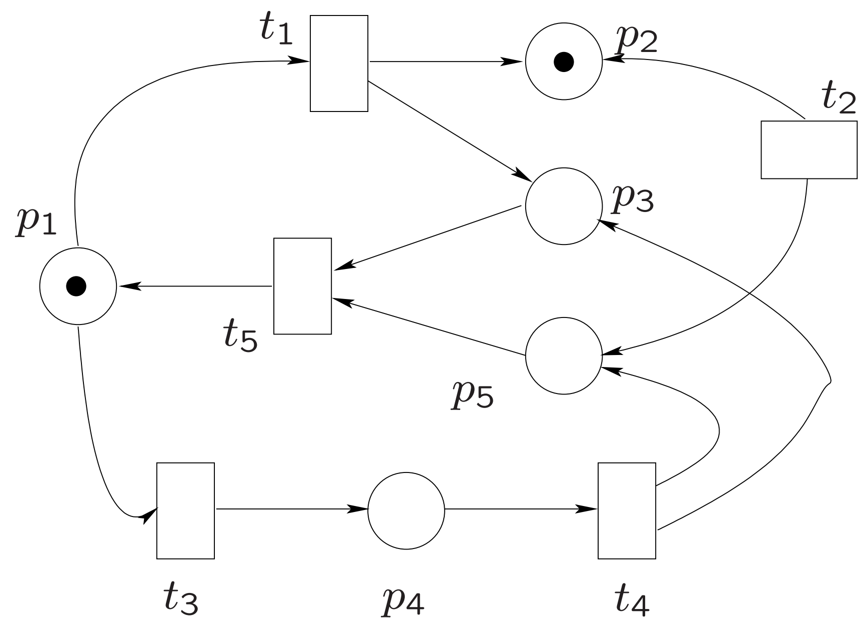



**Fig. 93.** p-invariants $\lambda \cdot (1,1,0,1,1) + \mu \cdot (1,0,1,1,0)$.

Its p-invariants are the solutions $i = (x_1, x_2, x_3, x_4, x_5)$ of:

$$\begin{array}{rrrrrl} -x_1 & + x_2 & + x_3 & & & = 0 \; , \\ & - x_2 & & & + x_5 & = 0 \; , \\ -x_1 & & & + x_4 & & = 0 \; , \\ & & + x_3 & - x_4 & + x_5 & = 0 \; , \\ +x_1 & & - x_3 & & - x_5 & = 0 \; . \end{array}$$

**Lemma 195.** Let $M$ be a P/T system and let $i$ be a positive p-invariant of $M$.
For $S = \{p \in P_M \mid i(p) > 0\}$, ${}^\bullet S = S^\bullet$.

**Lemma 47.** Let $M = (P, T, F, C_{in})$ be an EN system and let $S \subseteq P$.

There exists a subsystem $M'$ of $M$ with $P_{M'} = S$
iff ${}^\bullet S = S^\bullet$.

**Theorem 196.** Let $M$ be a P/T system and let $i$ be a positive p-invariant of $M$.
For all $p \in P_M$, if $i(p) > 0$, then $p$ is bounded.

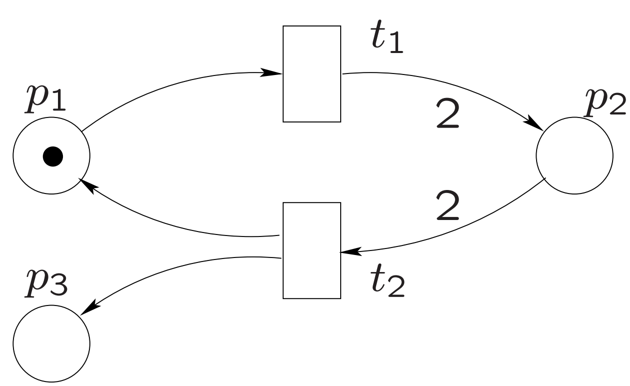
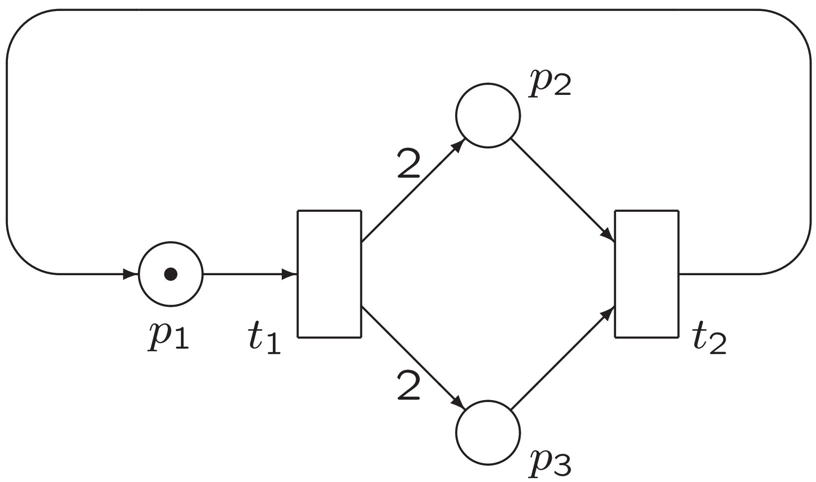



**Fig. 90.** P/T system $M''$ with invariant $2C(p_1) + C(p_2) = 2$.

**Theorem 196.** Let $M$ be a P/T system and let $i$ be a positive p-invariant of $M$.
For all $p \in P_M$, if $i(p) > 0$, then $p$ is bounded.

If $i$ is not a positive invariant, then the claim is not necessarily true:




$$i = (0, 1, -1) \text{ or } i = (0, -1, 1)$$

**Definition 197.** Let $M = (P, T, F, W, C_{in})$ be a P/T system.

A collection $\{i_1, \ldots, i_n\}$ of positive p-invariants of $M$ is a *covering* of $M$ if
for every place $p \in P$ there exists $k$, $1 \leq k \leq n$, with $i_k(p) > 0$.

Then $M$ is *covered by* $\{i_1, \ldots, i_n\}$.

**Lemma 195.** Let $M$ be a P/T system and let $i$ be a positive p-invariant of $M$.
For $S = \{p \in P_M \mid i(p) > 0\}$, ${}^\bullet S = S^\bullet$.

**Lemma 53.** Let $M = (P, T, F, C_{in})$ be an EN system, and let, for every $1 \leq i \leq n$ (with $n \geq 0$), $M_i = (S_i, T_i, F_i, (C_{in})_i)$ be a subsystem of $M$.

Then $\{M_1, \ldots, M_n\}$ is a covering of $M$
iff $P = \bigcup_{i=1}^{n} S_i$.

**Theorem 198.** If a P/T system $M$ is covered by positive p-invariants, then $M$ is bounded.

**Lemma 199.** A P/T system $M$ is covered by positive p-invariants iff there exists a p-invariant $i$ of $M$ such that $i(p) > 0$ for all $p \in P_M$.

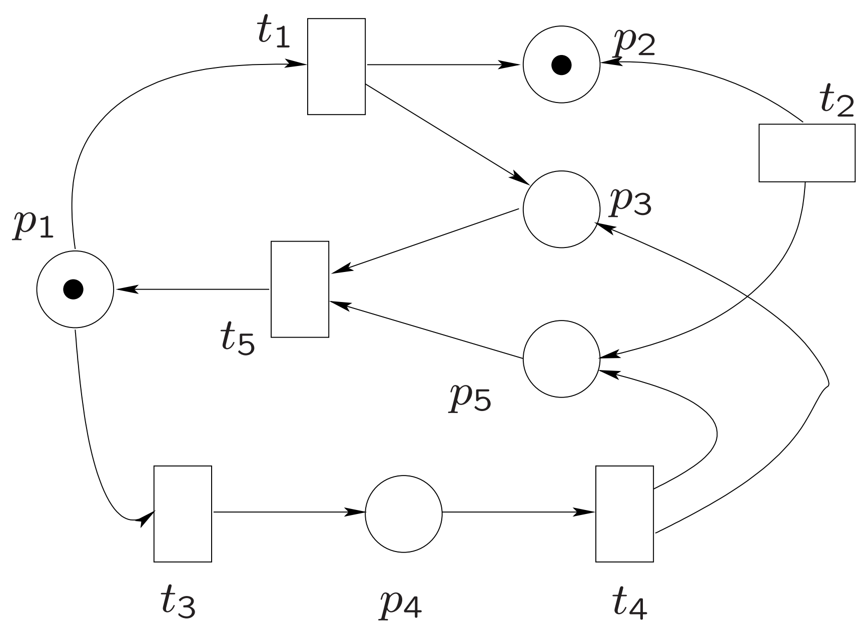



**Fig. 93.** p-invariants $\lambda \cdot (1,1,0,1,1) + \mu \cdot (1,0,1,1,0)$.

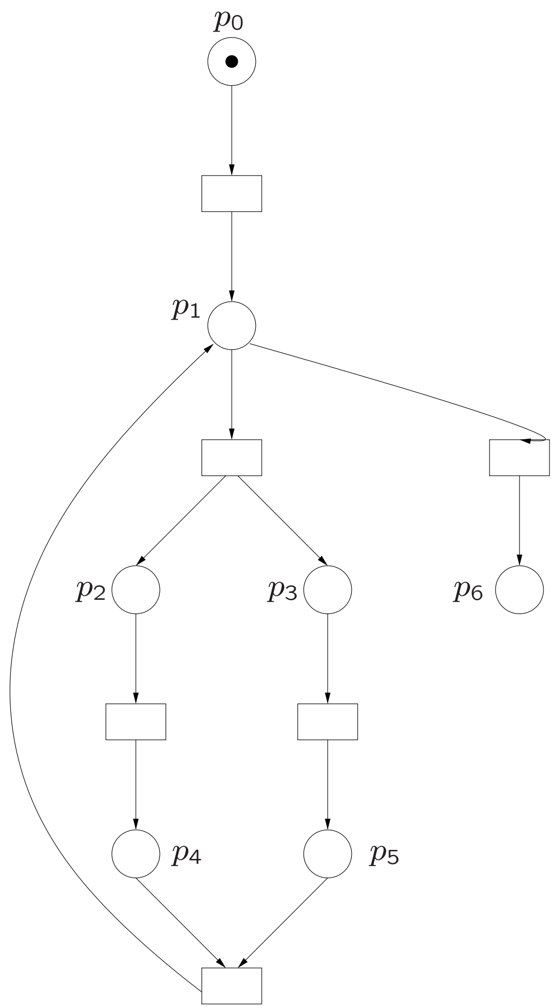



**Fig. 15.** A (contact-free) EN system, covered by sequential components $\{p_0, p_1, p_2, p_4, p_6\}$ and $\{p_0, p_1, p_3, p_5, p_6\}$.

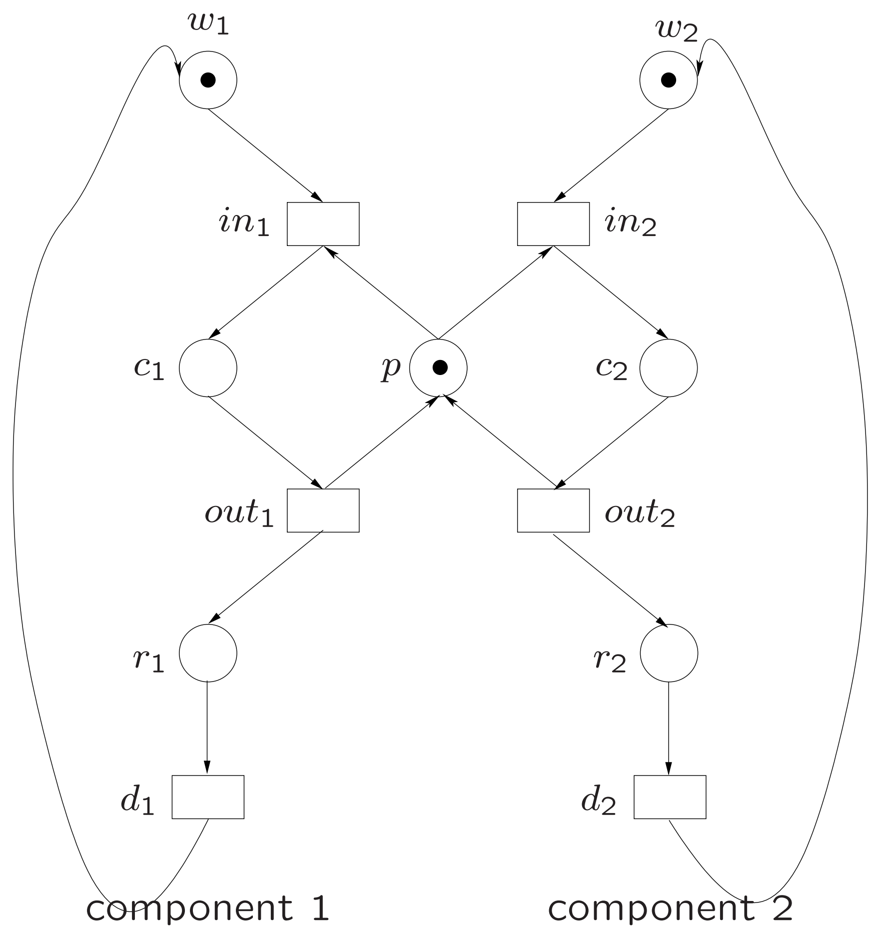



**Fig. 5.** The mutual exclusion problem.
A contact-free EN system, covered by sequential components $\{w_1, c_1, r_1\}$, $\{w_2, c_2, r_2\}$ and $\{p, c_1, c_2\}$.

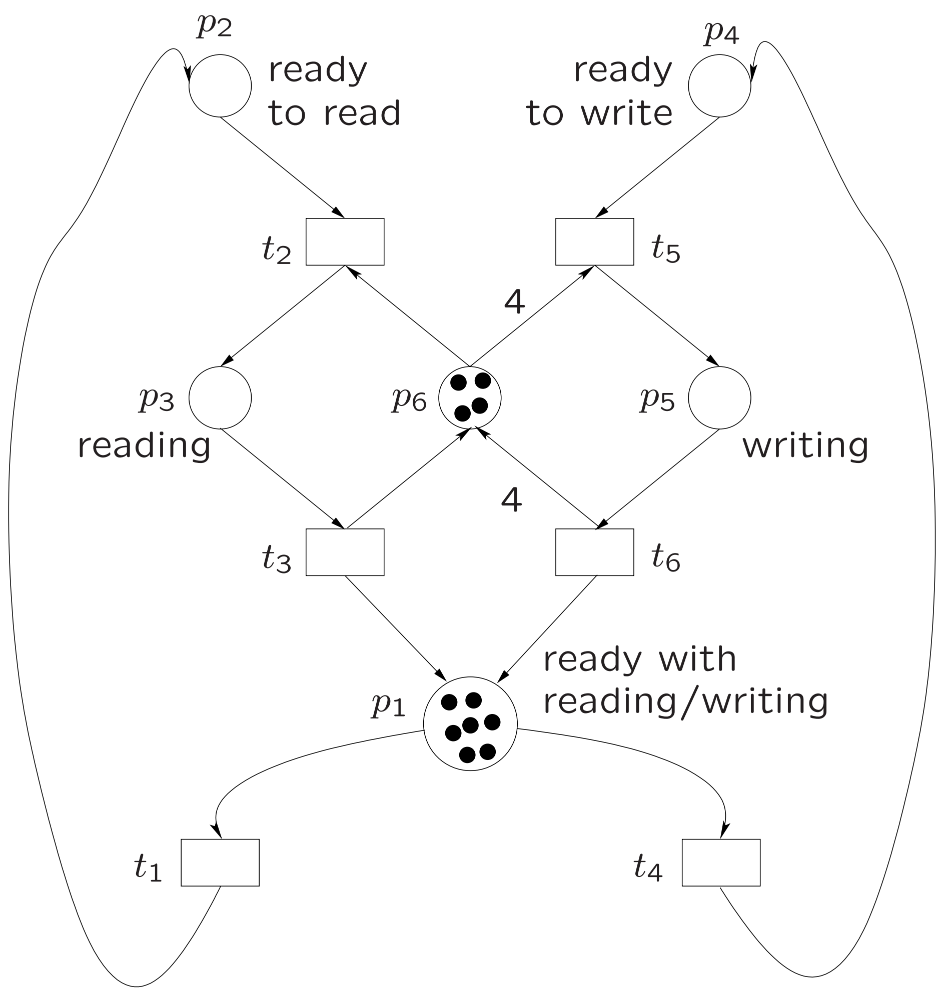



**Fig. 94.** An application: readers and writers.

$$
\begin{array}{rrrrrrrrrrrrrr}
- & x_1 & + & x_2 & & & & & & & & & = & 0 \\
 & & - & x_2 & + & x_3 & & & & & - & x_6 & = & 0 \\
+ & x_1 & & & - & x_3 & & & & & + & x_6 & = & 0 \\
- & x_1 & & & & & + & x_4 & & & & & = & 0 \\
 & & & & & & - & x_4 & + & x_5 & - & 4x_6 & = & 0 \\
+ & x_1 & & & & & & & - & x_5 & + & 4x_6 & = & 0
\end{array}
$$

yields the two p-invariants $i_1 = (1, 1, 1, 1, 1, 0)$ and $i_2 = (0, 0, 1, 0, 4, 1)$.

**Definition 200.** A P/T system $M$ is *live* if all transitions of $M$ are live.

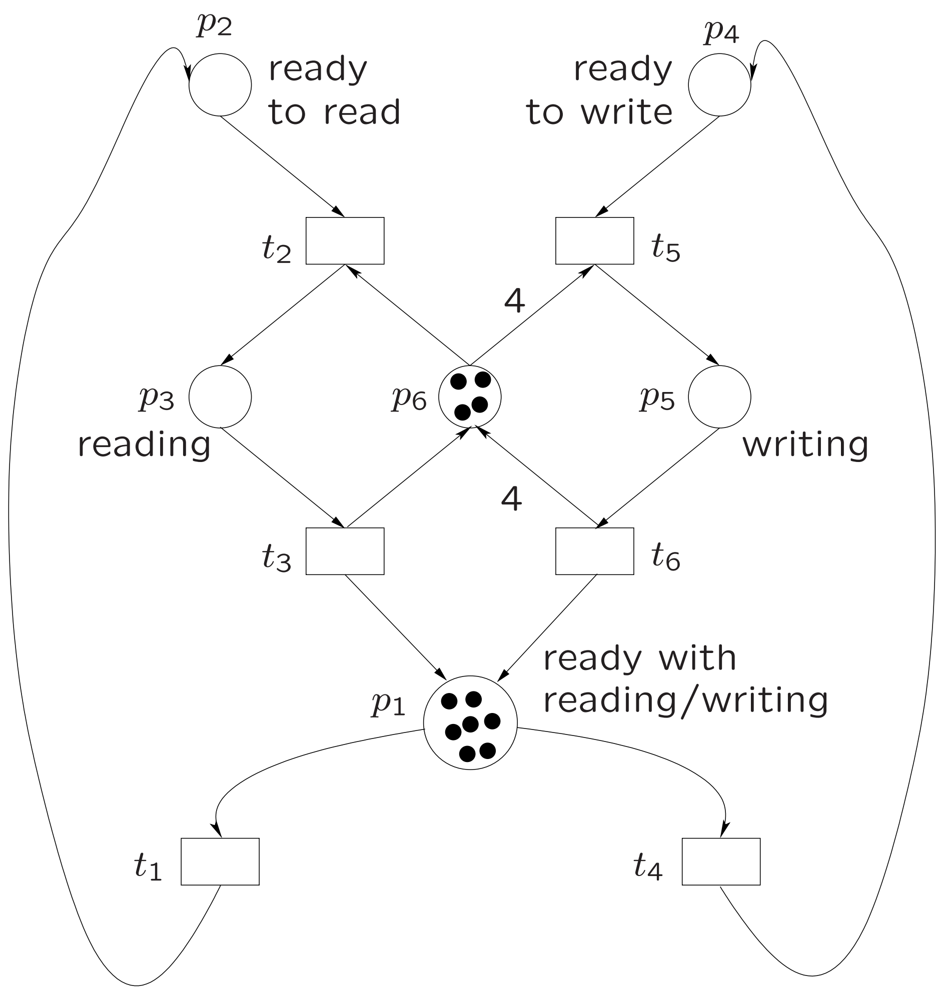



**Fig. 94.** Readers and writers, with the two p-invariants $i_1 = (1, 1, 1, 1, 1, 0)$ and $i_2 = (0, 0, 1, 0, 4, 1)$.

# 9.6. Marked Graphs

**Definition 201.** A *marked graph* (or *T-system*) is a P/T system $M = (P, T, F, W, C_{in})$, where

(1) $W(x, y) = 1$ for all $(x, y) \in F$, and

(2) $\#({}^\bullet p) = \#(p^\bullet) = 1$ for all $p \in P$.

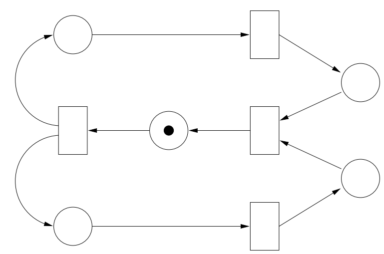


**Fig. 95.** A marked graph.

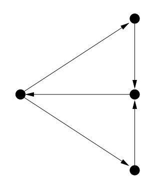


**Fig. 96.** The corresponding graph (without tokens).

**Definition 202.** Let $M = (P, T, F, W, C_{in})$ be a marked graph.

(1) A sequence $\alpha = (p_0, p_1, \ldots, p_m)$ of places is a *path of* $M$ *from* $p_0$ *to* $p_m$, if
$p_i^\bullet = {}^\bullet p_{i+1}$ for every $0 \leq i \leq m-1$.
If $p_m^\bullet = {}^\bullet p_0$ and all $p_i^\bullet$, $0 \leq i \leq m$, are different, then $\alpha$ is a (*elementary*) *cycle* of $M$;
$t \in T$ *is on* $\alpha$ if $t = p_i^\bullet$ for some $p_i$.

(2) $C(\alpha) = \sum_{i=0}^{m} C(p_i)$ is the *value of* $\alpha$ *in* $C$.
If $C(\alpha) > 0$, then $\alpha$ is *marked by* $C$.

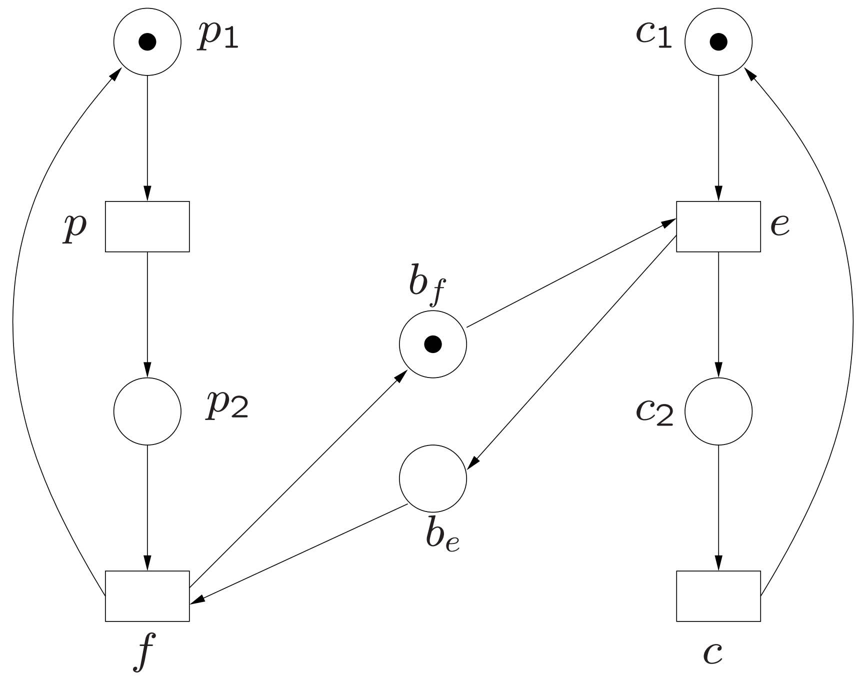



**Fig. 49.** The result of complementing $b = b_f$ in the producer/consumer system of Fig. 12.
A marked graph with three cycles.

**Lemma 203.** Let $M = (P, T, F, W, C_{in})$ be a marked graph. Let $\alpha = (p_0, \ldots, p_m)$ be a cycle of $M$ and let
$C_\alpha$ be the characteristic function of $\{p_0, \ldots, p_m\}$. Then

(1) $C_\alpha$ is a characteristic p-invariant of $M$, and

(2) $C(\alpha) = C_{in}(\alpha)$ for all $C \in \mathbb{C}_M$.

**Corollary 193.** Let $M = (P, T, F, W, C_{in})$ be a P/T system and let $i : P \to \mathbb{N}$ be the characteristic function of $S \subseteq P$.

Then $i$ is a (characteristic) p-invariant iff

$\sum\{W(p,t) \mid p \in {}^\bullet t \cap S\} = \sum\{W(t,p) \mid p \in t^\bullet \cap S\}$ for all $t \in T$.

If, in particular, $W(x,y) = 1$ for all $(x,y) \in F$, then this is equivalent with: $\#({}^\bullet t \cap S) = \#(t^\bullet \cap S)$ for all $t \in T$.

**Theorem 204.** Let $M = (P, T, F, W, C_{in})$ be a marked graph. Then

(1) $M$ is live,
iff
(2) $M$ is (strongly!) reduced,
iff
(3) all cycles of $M$ have a value $> 0$.

The proof of the implication (3)$\Rightarrow$(1) does not have to be known for the exam

**Theorem 205.** Let $M$ be a reduced marked graph. Then

(1) $M$ is safe,
iff
(2) $M$ is covered by sequential components,
iff
(3) every place of $M$ belongs to a cycle of $M$ with value 1.

**Corollary 206.** A marked graph $M$ is live and safe iff every cycle of $M$ is marked by $(C_{in})_M$ and every place of $M$ belongs to at least one cycle with value 1 in $(C_{in})_M$.

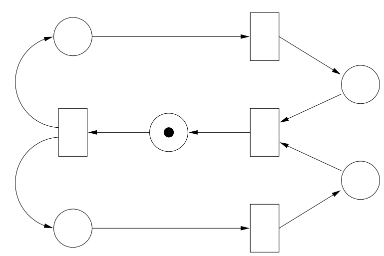


**Fig. 95.** A marked graph: live and safe.

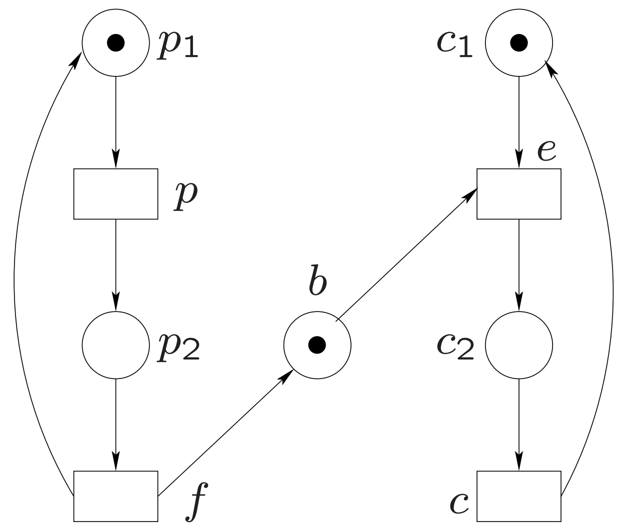



**Fig. 12.** The producer/consumer problem (to be considered as a marked graph): live and not safe.

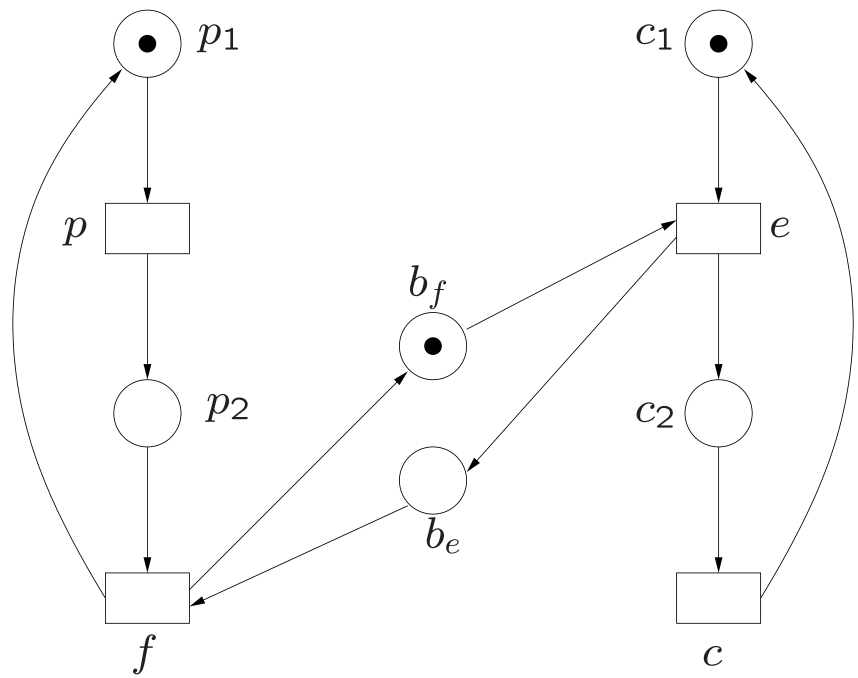



**Fig. 49.** The result of complementing $b = b_f$ in the producer/consumer system of Fig. 12: live and safe.

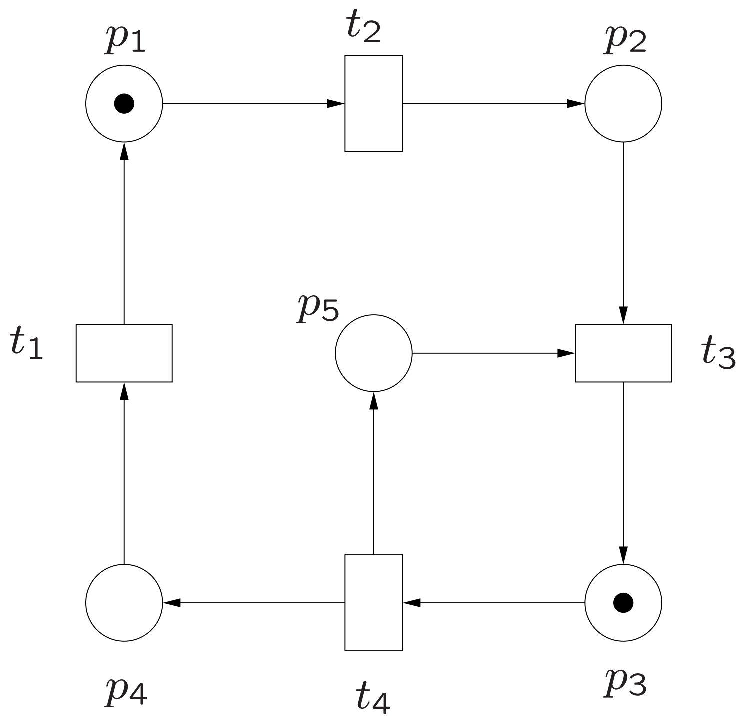



**Fig. 39.** A marked graph: live and not safe.

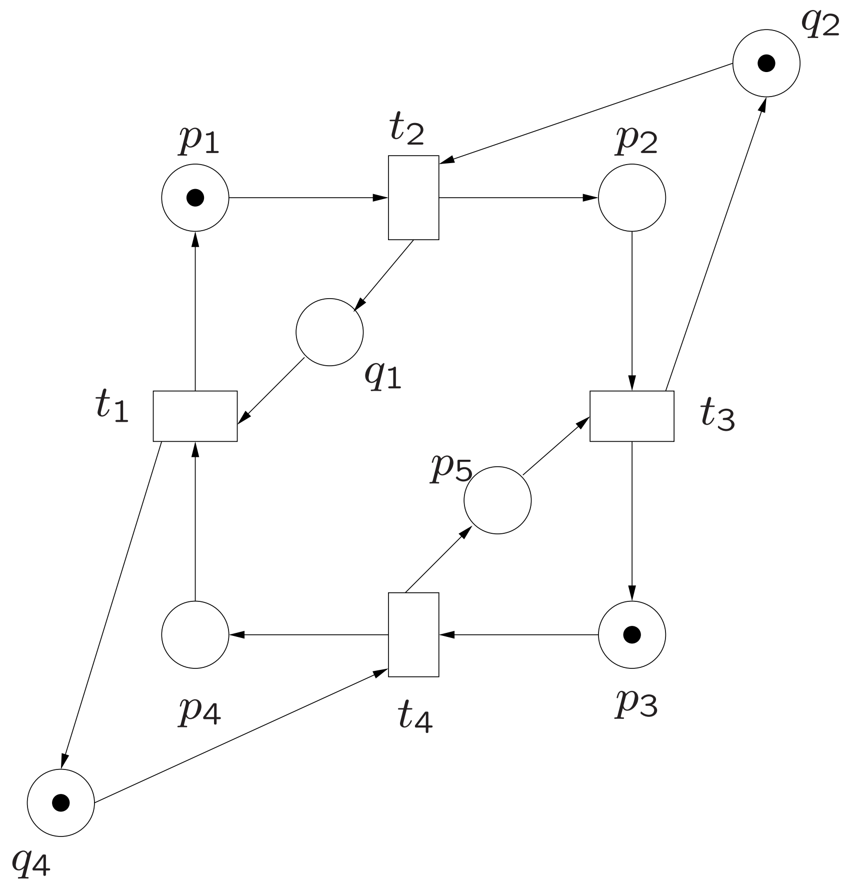



**Fig. 48.** The result of complementing $p_1$, $p_2$, $p_4$ in Fig. 39: live and safe.

**Theorem 59.** If a reduced EN system $M$
is covered by sequential components,
then $M$ is contact-free.
(but not the other way round!)

**Corollary 207.** Let $M$ be a reduced EN system which, moreover,
is a marked graph. Then
$M$ is contact-free iff
$M$ is covered by sequential components.

**Theorem 208.** Let $M$ be a marked graph.
Then $M$ can be started with a live and safe configuration iff each place of $M$ belongs to a cycle.

The proof from right to left does not have to be known for the exam

# 9.7. Free-Choice Systems

**Definition 209.** A *free-choice system* is a P/T system $M = (P, T, F, W, C_{in})$ such that

(1) for all $(x, y) \in F$, $W(x, y) = 1$, and

(2) for all $(p, t) \in F \cap (P \times T)$, $p^\bullet = \{t\}$ or ${}^\bullet t = \{p\}$.

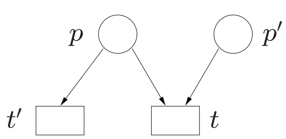


**Fig. 97.** Forbidden in free-choice systems.

A marked graph is always a free-choice system.

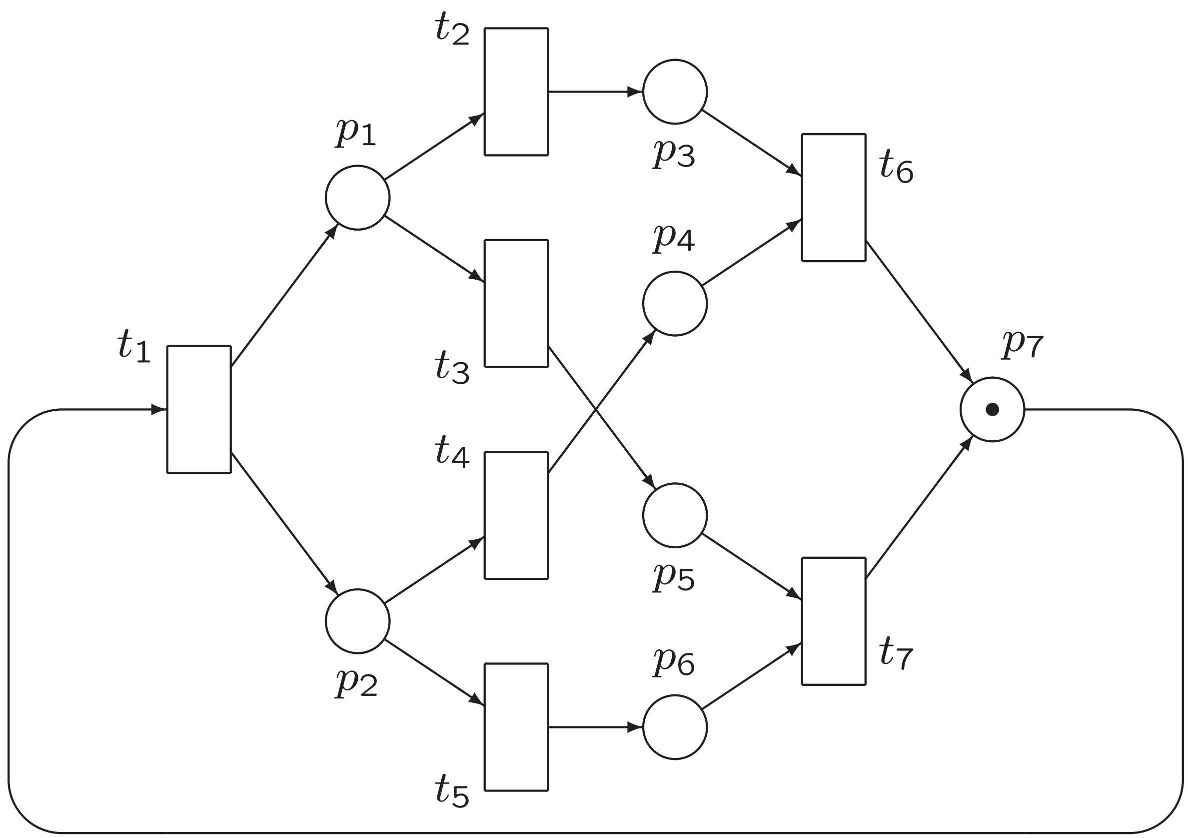



**Fig. 98.** A free-choice system.

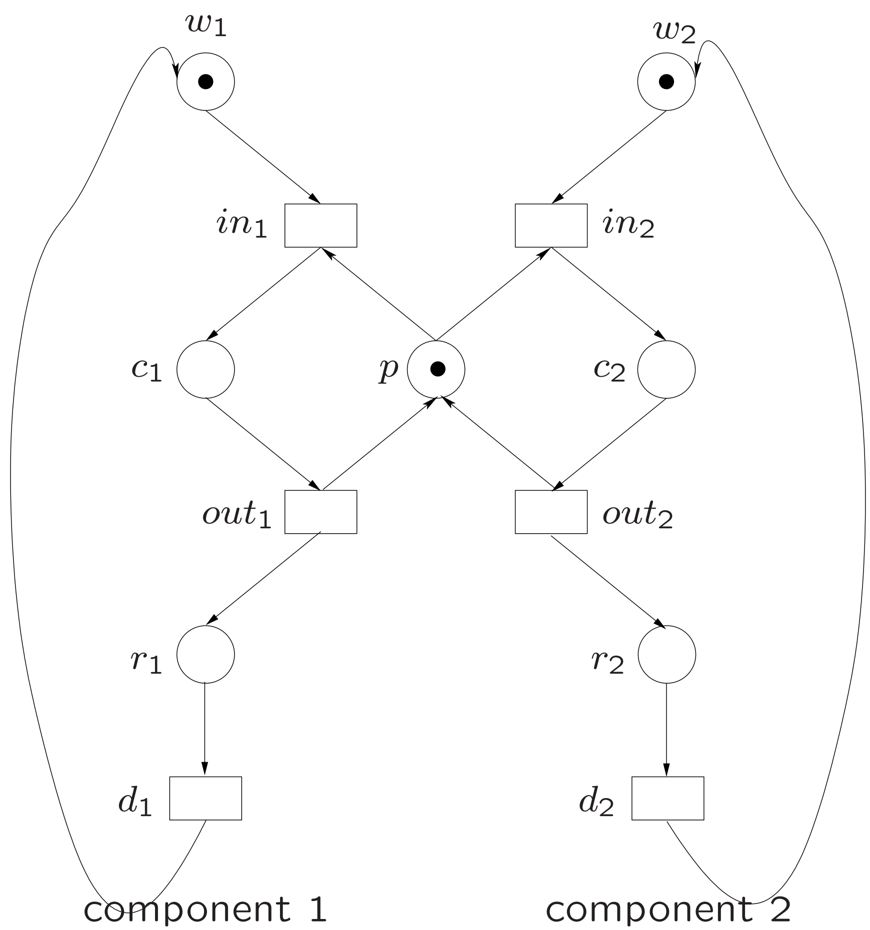



**Fig. 5.** The mutual exclusion problem: not free-choice (and not confusion-free).

**Definition 210.** Let $M$ be a P/T system and let $S \subseteq P_M$.

(1) $S$ is a *siphon* if ${}^\bullet S \subseteq S^\bullet$.

(2) $S$ is a *trap* if $S^\bullet \subseteq {}^\bullet S$.

**Theorem 211.** Let $M = (P, T, F, W, C_{in})$ be a P/T system and let $S \subseteq P$.

(1) If $S$ is a siphon and $C_{in}(p) = 0$ for every $p \in S$, then $C(p) = 0$ for every $C \in \mathbb{C}_M$ and every $p \in S$.

(2) If $S$ is a trap and there is a $p \in S$ with $C_{in}(p) > 0$, then for every $C \in \mathbb{C}_M$ there is a $p' \in S$ with $C(p') > 0$.

**Theorem 212.** Let $M$ be a free-choice system without isolated places. Then

$M$ is live iff
every nonempty siphon contains a trap
that has at least one token in $(C_{in})_M$.

Without a proof

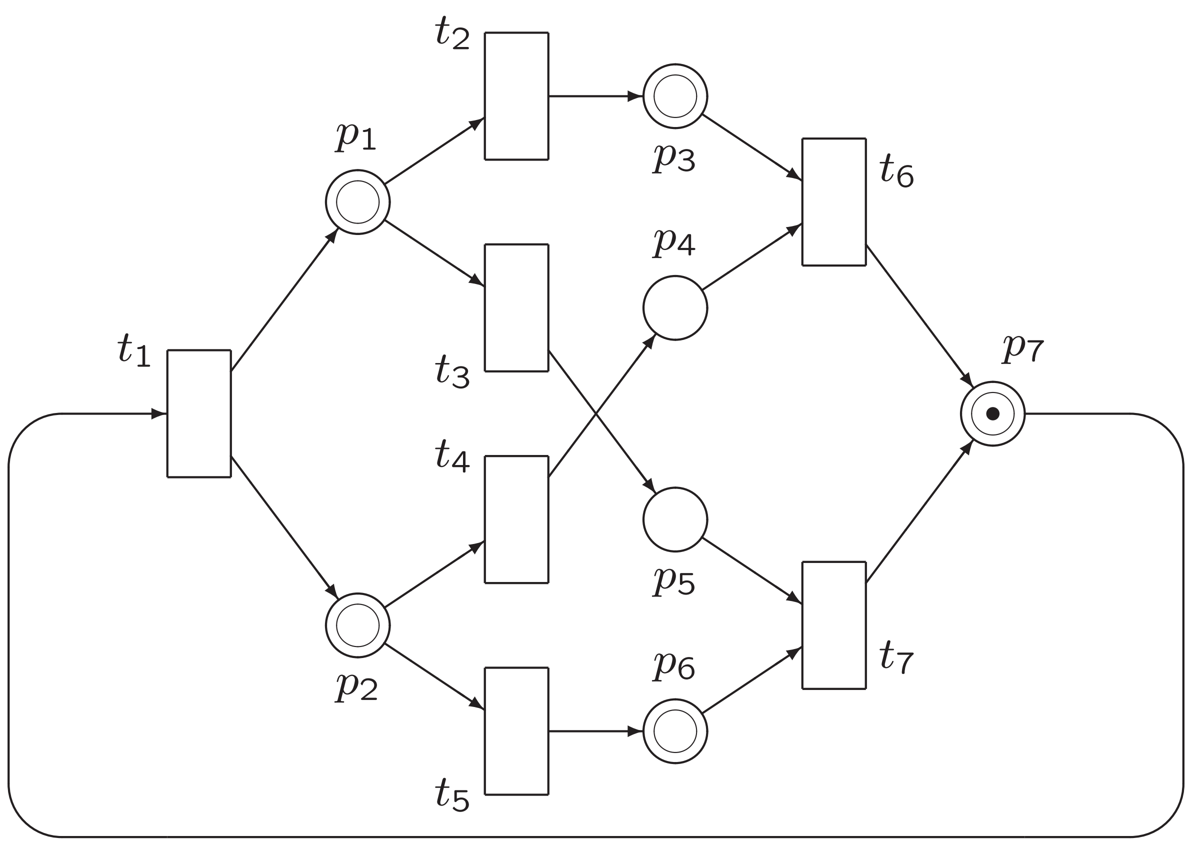



**Fig. 98.** A free-choice system with a syphon.

**Theorem 213.** Let $M$ be a live free-choice system without isolated places. Then

$M$ is safe iff
$M$ is covered by sequential components.

Without a proof