Logica (I&E)

najaar 2018

`http://liacs.leidenuniv.nl/~vlietrvan1/logica/`

Rudy van Vliet
kamer 140 Snellius, tel. 071-527 2876
rvvliet(at)liacs(dot)nl

college 10, maandag 12 november 2018

2. Predicate logic
2.2. Predicate logic as a formal language
2.3. Proof theory of predicate logic

Hun verdediging was geitenkaas.

A slide from lecture 9:

2.2.1. Terms

Definition 2.1. Terms over $\mathcal{F}$ are defined as follows.

- Any variable is a term.
- If $c \in \mathcal{F}$ is a nullary function, then c is a term.
- If $t_1, t_2, \ldots, t_n$ are terms and $f \in \mathcal{F}$ has arity $n > 0$, then $f(t_1, t_2, \ldots, t_n)$ is a term.
- Nothing else is a term.

Dependent on set $\mathcal{F}$

A slide from lecture 9:

Formulas

Definition 2.3. Formulas over $(\mathcal{F}, \mathcal{P})$ are defined as follows.

- If $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 0$, and if $t_1, t_2, \ldots, t_n$ are terms over $\mathcal{F}$, then $P(t_1, t_2, \ldots, t_n)$ is a formula.
- If ϕ is a formula, then so is $(\neg\phi)$
- If ϕ and ψ are formulas, then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$ and $(\phi \rightarrow \psi)$.
- If ϕ is a formula and x is a variable, then $(\forall x \phi)$ and $(\exists x \phi)$ are formulas.
- Nothing else is a formula.

Some intuition. . .

A slide from lecture 9:

Definition 2.6. Let ϕ be a formula in predicate logic.
An occurrence of x in ϕ is free in ϕ if it is a leaf node in the parse tree of ϕ such that there is no path upwards from that node x to a node $\forall x$ or $\exists x$.
Otherwise, that occurrence of x is called bound.
For $\forall x \phi$ or $\exists x \phi$, we say that ϕ – minus any of ϕ's subformulas $\exists x \psi$ or $\forall x \psi$ – is the scope of $\forall x$, respectively $\exists x$.

A slide from lecture 9:

Substitution

Variables are placeholders

Definition 2.7.
Given a variable x, a term t and a formula ϕ, we define $\phi[t/x]$ to be the formula obtained by replacing each free occurrence of variable x in ϕ with t.

$\phi[t/x] \ldots$

Example 2.11.

$$\phi = \exists y(x < y)$$

$\forall x \phi$ . . .

parse tree. . .

$\phi[y/x]$ . . .

Definition 2.8.
Given a term t, a variable x and a formula ϕ,
we say that t is free for x in ϕ,
if no free x leaf in ϕ occurs in the scope of $\forall y$ or $\exists y$ for any variable y occurring in t.

Definition 2.8.
Given a term t, a variable x and a formula ϕ,
we say that t is free for x in ϕ,
if no free x leaf in ϕ occurs in the scope of $\forall y$ or $\exists y$ for any variable y occurring in t.

Example 2.9.
Parse tree of

$$\phi = S(x) \wedge \forall y(P(x) \rightarrow Q(y))$$

$\phi[f(y,y)/x] \ldots$

Definition 2.8.
Given a term t, a variable x and a formula ϕ, we say that t is free for x in ϕ, if no free x leaf in ϕ occurs in the scope of $\forall y$ or $\exists y$ for any variable y occurring in t.

If no free occurrences of x in ϕ...

If t is not free for x in ϕ...

If t equals x...

2.3. Proof theory of predicate logic

2.3.1. Natural deduction rules

Extra rules

The proof rules for equality

In terms of computation results

$$\forall x \forall y \forall u \forall v (M(x, y) \land M(y, a) \land M(u, v) \land M(v, p) \to x = u)$$

Equality introduction

$$\frac{}{t = t} = \mathsf{i}$$

Sound

Equality elimination

$$\frac{t_1 = t_2 \qquad \phi[t_1/x]}{\phi[t_2/x]} = \mathsf{e}$$

Equality elimination

$$\frac{t_1 = t_2 \quad \phi[t_1/x]}{\phi[t_2/x]} = \mathsf{e}$$

Convention 2.10.
When we write a substitution in the form $\phi[t/x]$, we implicitly assume that t is free for x in ϕ.

Example.

$x+1 = 1+x,\ (x+1 > 1) \to (x+1 > 0) \ \vdash\ (1+x > 1) \to (1+x > 0)$

Proof. . .

$\phi = \ldots$

Sound

Example.

$x{+}1 = 1{+}x,\ (x{+}1 > 1) \to (x{+}1 > 0) \vdash (1{+}x > 1) \to (1{+}x > 0)$

Proof:

1	$x + 1 = 1 + x$	premise
2	$(x + 1 > 1) \to (x + 1 > 0)$	premise
3	$(1 + x > 1) \to (1 + x > 0)$	$=$e 1,2, ϕ: $(y > 1) \to (y > 0)$

Example.

$x+1 = 1+x,\ (x+1 > 1) \to (x+1 > 0) \vdash (1+x > 1) \to (1+x > 0)$

Proof:

1	$x + 1 = 1 + x$	premise
2	$(x + 1 > 1) \to (x + 1 > 0)$	premise
3	$(1 + x > 1) \to (1 + x > 0)$	$=$e 1,2, ϕ: $(x > 1) \to (x > 0)$

$$\frac{t_1 = t_2 \quad \phi[t_1/x]}{\phi[t_2/x]} = \mathsf{e}$$

We do not demand that ϕ is true,
we demand that $\phi[t_1/x]$ is true

Example.

$$t_1 = t_2 \;\vdash\; t_2 = t_1$$

Proof. . .

$\phi = \ldots$

Example.

$$t_1 = t_2, t_2 = t_3 \ \vdash \ t_1 = t_3$$

Proof. . .

$\phi = \ldots$

Reflexive:

$$\frac{}{t = t} \; =\mathsf{i}$$

Symmetric:

$$t_1 = t_2 \;\vdash\; t_2 = t_1$$

Transitive:

$$t_1 = t_2, t_2 = t_3 \;\vdash\; t_1 = t_3$$

Universal quantification elimination

$$\frac{\forall x \phi}{\phi[t/x]} \ \forall x \text{ e}$$

Sound

Example.

$$P(t),\ \forall x(P(x) \to \neg Q(x))\ \vdash\ \ldots$$

Example.

$$P(t),\ \forall x(P(x) \rightarrow \neg Q(x))\ \vdash\ \neg Q(t)$$

Proof . . .

Universal quantification introduction

Let x_0 be fresh variable

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad\vdots \\ \qquad \phi[x_0/x] \end{array}}}{\forall x \phi}\ \forall x \text{ i}$$

Example.

$$\forall x(P(x) \to Q(x)), \forall x P(x) \ \vdash \ \forall x Q(x)$$

Proof . . .

Setup of proof . . .

$\forall x$ e	$\forall x$ i
$\forall y$ e	$\forall y$ i
$\forall$e	$\forall$i

Analogy $\wedge$ and $\forall$

elimination		introduction
$\dfrac{\phi \wedge \psi}{\phi}\ \wedge e_R$	$\dfrac{\phi \wedge \psi}{\psi}\ \wedge e_L$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge i$
$\dfrac{\forall x \phi}{\phi[t/x]}\ \forall x\ e$		$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad \vdots \\ \quad \phi[x_0/x] \end{array}}}{\forall x \phi}\ \forall x\ i$

Analogy $\vee$ and $\exists$

elimination	introduction
$$\dfrac{\phi\vee\psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$$	$$\dfrac{\phi}{\phi\vee\psi}\ \vee\mathrm{i}_R \qquad \dfrac{\psi}{\phi\vee\psi}\ \vee\mathrm{i}_L$$
	...

Analogy $\vee$ and $\exists$

elimination	introduction
$$\dfrac{\phi\vee\psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$$	$$\frac{\phi}{\phi\vee\psi}\ \vee\mathrm{i}_R \qquad \frac{\psi}{\phi\vee\psi}\ \vee\mathrm{i}_L$$
...	$$\frac{\phi[t/x]}{\exists x\phi}\ \exists x\ \mathrm{i}$$

Notation $\phi[t/x]$ in $\exists x$ i...

Analogy $\vee$ and $\exists$

elimination	introduction
$$\dfrac{\phi\vee\psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\text{e}$$	$$\dfrac{\phi}{\phi\vee\psi}\ \vee\text{i}_R \qquad \dfrac{\psi}{\phi\vee\psi}\ \vee\text{i}_L$$
$$\dfrac{\exists x\phi \quad \boxed{\begin{array}{cc}x_0 & \phi[x_0/x]\\ & \vdots\\ & \chi\end{array}}}{\chi}\ \exists x\ \text{e}$$	$$\dfrac{\phi[t/x]}{\exists x\phi}\ \exists x\ \text{i}$$

Example.

$$\forall x \phi \ \vdash \ \exists x \phi$$

Proof. . .

Example.

$$\forall x(P(x) \rightarrow Q(x)), \exists x P(x) \ \vdash \ \ldots$$

Example.

$$\forall x(P(x) \rightarrow Q(x)), \exists x P(x) \;\vdash\; \exists x Q(x)$$

Proof. . .

Example.

$$\forall x(P(x) \to Q(x)), \exists x P(x) \;\vdash\; \exists x Q(x)$$

Alternative ‘proof’. . .

Example.

$$\forall x(P(x) \to Q(x)), \exists x P(x) \ \vdash \ \exists x Q(x)$$

Alternative 'proof' (illegal):

1	$\forall x(P(x) \to Q(x))$	premise
2	$\exists x P(x)$	premise
3	$x_0 : P(x_0)$	assumption
4	$P(x_0) \to Q(x_0)$	$\forall x$ e 1
5	$Q(x_0)$	$\to$ e 4,3
6	$Q(x_0)$	$\exists x$ e 2, 3–5
7	$\exists x Q(x)$	$\exists x$ i 6

Example.

$$\forall x(Q(x) \to R(x)),\ \exists x(P(x) \wedge Q(x)) \ \vdash\ \exists x(P(x) \wedge R(x))$$

Proof. . .

Study this example yourself.