LIACS | Examination Logic
Leiden University | 8 July 2016, 10:00 - 13:00

1. **[1 point]** Draw the parse tree of the formula $p \to ((q \wedge \neg\neg p) \vee \neg(q \to p))$ and list *all* its subformulas.

2. **[2 points]** Give a proof by means of natural deduction of the following sequents:

a) $\vdash p \to ((p \to q) \to q)$.

1	p	assumption
2	$p \to q$	assumption
3	q	$\to$e 1,2
4	$(p \to q) \to q)$.	$\to$i 2-3
5	$p \to ((p \to q) \to q)$	$\to$i 1-4

b) $\neg p \vdash p \to (p \to q)$.

1	$\neg p$	premise
2	p	assumption
3	$\bot$	$\neg$e 1,2
4	$p \to q$	$\bot$e 3
5	$p \to (p \to q)$	$\to$i 2-6

c) $(p \to q) \vee (r \to q) \vdash (p \wedge r) \to q$

1	$(p \to q) \vee (r \to q)$	premise		
2	$p \wedge r$	assumption		
3	$p \to q$	assumption	$r \to q$	assumption
4	p	$\wedge$e 2	r	$\wedge$e 2
5	q	$\to$e 4, 3	q	$\to$e 4, 3
6	q	$\vee$e 1, 3-5		
7	$(p \wedge r) \to q$	$\to$i 2, 6		

d) $\neg p, (p \vee q) \vdash q$.

1	$\neg p$	premise		
2	$p \vee q$	premise		
3	p	assumption	q	assumption
4	$\bot$	$\neg$e 1,3		
5	q	$\bot$e 4		
6	q	$\vee$e 2,3-5,3		

3. **[1 point]** Use mathematical induction to prove that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all $n \geq 1$.

Proof:

Let $n = 1$. Then the left hand side is $\frac{1}{1(1+1)} = \frac{1}{2}$ which is clearly equal to the right hand. Thus the statement we need to prove holds for $n = 1$.

Assume now the statement holds for $n = k$; that is, $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$, and let us consider the case when $n = k + 1$:

$$\sum_{i=1}^{k+1}\frac{1}{i(i+1)}=\sum_{i=1}^{k}\frac{1}{i(i+1)}+\frac{1}{(k+1)(k+2)}$$ splitting the sum in two parts

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$ here we use the induction hypothesis!

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$ algebraic calculation

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$ algebraic calculation

$$=\frac{k+1}{(k+2)}$$ algebraic calculation

qed

4. **[2 points]** Compute the conjunctive normal form of the following formulas and check which formulas are valid. Explain your answer.

a) $(p \wedge \neg q) \vee (p \wedge q)$.

We have $(p \wedge \neg q) \vee (p \wedge q) \equiv (p \vee (p \wedge q)) \wedge (\neg q \vee (p \wedge q))$ (distributive laws)

$\equiv (p \vee p) \wedge (p \vee q) \wedge (\neg q \vee p) \wedge (\neg q \vee q)$ (distributive laws)

Since the first three conjuncts are not valid, the entire formula is not valid.

b) $\neg(p \wedge \neg q) \wedge (q \vee \neg p)$.

We have $\neg(p \wedge \neg q) \wedge (q \vee \neg p) \equiv (\neg p \vee \neg\neg q) \wedge (q \vee \neg p)$ (De Morgan's laws)

$\equiv (\neg p \vee q) \wedge (q \vee \neg p)$ (double negation)

Since the first conjunct is not valid, the entire formula is not valid.

c) $((p \to q) \vee p) \wedge (p \vee \neg(r \wedge \neg r \wedge q))$.

We have $((p \to q) \vee p) \wedge (p \vee \neg(r \wedge \neg r \wedge q)) \equiv ((\neg p \vee q) \vee p) \wedge (p \vee \neg(r \wedge \neg r \wedge q))$ (implication)

$\equiv (\neg p \vee q \vee p) \wedge (p \vee (\neg r \vee \neg\neg r \vee \neg q))$ (De Morgan)

$\equiv (\neg p \vee q \vee p) \wedge (p \vee \neg r \vee r \vee \neg q)$ (double negation)

Since the both conjuncts are valid, the entire formula is valid.

d) Construct a formula ϕ in conjunctive normal form from the truth table

p	q	ϕ
T	T	F
T	F	T
F	T	T
F	F	F

The formula ϕ is obtained as the conjunction of the disjunction of the opposite atoms of the line where ϕ is false: $(\neg p \vee \neg q) \wedge (p \vee q)$

5. **[1 point]** Apply the marking algorithm to check if the following Horn formulas are satisfiable:

a) $(\top \to p) \wedge ((p \wedge q) \to r) \wedge (p \to q) \wedge ((r \wedge p) \to q)$.

Let us mark the propositions by using subscripts indicating the marking round. We have

$$(\top_1 \to p_2) \wedge ((p_2 \wedge q_3) \to r_4) \wedge (p_2 \to q_3) \wedge ((r_4 \wedge p_2) \to q_3) .$$

Thus the formula is satisfiable under any valuations mapping p,q and r to T.

b) $(\top \to p) \wedge (p \to q) \wedge ((p \wedge q) \to r) \wedge (q \to \bot) \wedge (\top \to r)$.

Let us mark the propositions by using subscripts indicating the marking round. We have

$$(\top_1 \to p_2) \wedge (p_2 \to q_3) \wedge ((p_2 \wedge q) \to r_2) \wedge (q_3 \to \bot_4) \wedge (\top_1 \to r_2)$$

Thus the formula is not satisfiable.

6. **[2 points]** Show the validity by means of natural deduction of the following sequents:

a) $\forall x P(x), \neg\exists x Q(x) \vdash P(a) \vee Q(a)$.

1	$\forall x P(x)$	premise

2	$\neg\exists x Q(x)$	premise
3	$P(a)$	$\forall$e 1
4	$P(a) \vee Q(a)$	$\vee i_L$ 3

b) $P(a) \vdash \forall x\,(x = a \to P(x))$.

1		$P(a)$	assumption
2	x_0		
3		$x_0 = a$	assumption
4		$P(x_0)$	=e 2,1
6		$x_0 = a \to P(x_0)$	$\to$i 3-4
7		$\forall x\,(x = a \to P(x))$	$\forall$i 2-6

c) $\vdash \exists x(x = a \vee \neg(x = b))$.

1	$a = a$	=i
2	$a = a \vee \neg(a = b)$	$\vee$i 1
3	$\exists x(x = a \vee \neg(x = b))$	$\exists$i 1-2

d) $\vdash \neg\exists x \neg(x = x)$.

1		$\exists x \neg(x = x)$	assumption
2	x_0	$\neg(x_0 = x_0)$	assumption
3		$x_0 = x_0$	=i
4		$\perp$	$\neg$e 2,3
5		$\perp$	$\exists$e 1,2-4
6		$\neg\exists x \neg(x = x)$	$\neg$i 1,5

7. **[1 point]** For each of the following sequents give a model showing that it is not valid:

a) $\vdash \forall x \forall y \forall z\,(P(x,y) \to P(y,z))$.

Take the model M with the set {0,1} as universe, and $P^M = \{(0,1)\}$. Then for the environment assigning x to 0, y to 1 and z to 0 we have that $P(x,y) \to P(y,z)$ does not hold.

b) $\forall x(P(x) \vee Q(x)) \vdash \forall x P(x) \vee \forall x Q(x)$.

Take the model M with the set {0,1} as universe, and $P^M = \{0\}$, $Q^M = \{1\}$. Then the right hand side clearly does not hold, while for every environment either P(x) or Q(x) holds.

The final score is given by the sum of the points obtained.