

From lecture 3:

Exercise 4.45. ♠

Show that the CFG below is unambiguous.

b. The CFG with productions $S \rightarrow (S)S \mid \Lambda$

Exercise.

Let G be a context-free grammar with start variable S and the following productions:

$$S \rightarrow aSbS \mid bSaS \mid \Lambda$$

- a. Show that $L(G) = AEqB = \{x \in \{a,b\}^* \mid n_a(x) = n_b(x)\}$. That is, argue why $L(G) \subseteq AEqB$ and why $AEqB \subseteq L(G)$. You do not have to give formal proofs.
- b. Show that G is ambiguous, by giving a string $x \in L(G)$ and two different derivation trees for x in G .
- c. ♣ Give an unambiguous context-free grammar for $AEqB$.

From lecture 2:

Example 4.10.

Find a context-free grammar generating the language

$$\{a^i b^j c^k \mid j \neq i + k\}$$

Hint: Consider the cases $j > i + k$ and $j < i + k$ separately.

Exercise. *From exam Automata Theory, 19 December, 2024*

Let

$$L = \{a^i b^j a^k \mid i, j, k \geq 0 \text{ and } j < i + k\}$$

- (a) Give the first six elements in the canonical (shortlex) order of L .
- (b) Give a context-free grammar G , such that $L(G) = L$. Try to ensure that G is unambiguous.

If your context-free grammar is ambiguous, then give two different derivation trees for a string $x \in L$.

Exercise 4.51. ♣

A variable A in a context-free grammar $G = (V, \Sigma, S, P)$ is *live* if $A \Rightarrow^* x$ for some $x \in \Sigma^*$. A variable that is not live, is called *dead*.

Give a recursive definition, and a corresponding algorithm, for finding all live variables in G .

Exercise 4.52. ♣

A variable A in a context-free grammar $G = (V, \Sigma, S, P)$ is *reachable* if $S \Rightarrow^* \alpha A \beta$ for some $\alpha, \beta \in (\Sigma \cup V)^*$.

Give a recursive definition, and a corresponding algorithm, for finding all reachable variables in G .

Exercise 4.53.

A variable A in a context-free grammar $G = (V, \Sigma, S, P)$ is *useful* if for some string $x \in \Sigma^*$, there is a derivation of x that takes the form $S \Rightarrow^* \alpha A \beta \Rightarrow^* x$ for some $\alpha, \beta \in (\Sigma \cup V)^*$.

A variable that is not useful is *useless*. Clearly, if a variable is either not live or not reachable (see the preceding two exercises), then it is useless. The converse is not true, as the grammar with productions $S \rightarrow AB$ and $A \rightarrow a$ illustrates. (The variable A is both live and reachable but still useless.)

a. ♣ Let G be a CFG. Suppose G_1 is obtained by eliminating all dead variables from G and all productions in which dead variables appear. Suppose G_2 is then obtained from G_1 by eliminating all variables unreachable in G_1 , as well as productions in which such variables appear. Show that G_2 contains no useless variables, and $L(G_2) = L(G)$.

Exercise 4.53. (continued)

- a. ♣** Let G be a CFG. Suppose G_1 is obtained by eliminating all dead variables from G and all productions in which dead variables appear. Suppose G_2 is then obtained from G_1 by eliminating all variables unreachable in G_1 , as well as productions in which such variables appear. Show that G_2 contains no useless variables, and $L(G_2) = L(G)$.
- b.** Give an example to show that if the two steps are done in the opposite order, the resulting grammar may still have useless variables.

Exercise 4.53. (continued)

c. In each case, given the context-free grammar G , find an equivalent CFG with no useless variables.

i. ♣ G has productions

$$\begin{array}{ll} S \rightarrow ABC \mid BaB & A \rightarrow aA \mid BaC \mid aaa \\ B \rightarrow bBb \mid a & C \rightarrow CA \mid AC \end{array}$$

ii. G has productions

$$\begin{array}{lll} S \rightarrow AB \mid AC & A \rightarrow aAb \mid bAa \mid a & B \rightarrow bbA \mid aaB \mid AB \\ C \rightarrow abCa \mid aDb & D \rightarrow bD \mid aC & \end{array}$$

Exercise 4.49. ♣

In each case below, find a context-free grammar with no Λ -productions that generates the same language, except possibly for Λ , as the given CFG.

a.

$$S \rightarrow AB \mid \Lambda \quad A \rightarrow aASb \mid a \quad B \rightarrow bS$$

Exercise. *From exam Automata Theory, 19 December, 2022*

Let G_1 be the context-free grammar with start variable S and the following productions:

$$S \rightarrow Sa \mid bb \mid AB \quad A \rightarrow aAb \mid BBa \quad B \rightarrow SB \mid a \mid \Lambda$$

- a. Determine the nullable variables in G_1 .
- b. Give the context-free grammar G_2 resulting from G_1 by eliminating Λ -productions.
- c. For each variable X in G_2 , give the set of X -derivable variables.
- d. Give the context-free grammar G_3 resulting from G_2 by eliminating unit productions.

Exercise 4.50.

In each case, given the context-free grammar G , find a CFG G' in **Chomsky normal form** that generates the language $L(G) - \{\Lambda\}$.

a. G has productions

$$S \rightarrow ABA \quad A \rightarrow aA \mid \Lambda \quad B \rightarrow bB \mid \Lambda$$

b. G has productions

$$S \rightarrow aSa \mid bSb \mid \Lambda \quad A \rightarrow aBb \mid bBa \quad B \rightarrow aB \mid bB \mid \Lambda$$

Exercise 4.54.

In each case below, given the context-free grammar G , find a CFG G_1 in Chomsky normal form generating $L(G) - \{\Lambda\}$.

a. ♣ G has productions $S \rightarrow SS \mid (S) \mid \Lambda$

b. G has productions $S \rightarrow S(S) \mid \Lambda$

c. G has productions

$$\begin{array}{ll} S \rightarrow AaA \mid CA \mid BaB & A \rightarrow aaBa \mid CDA \mid aa \mid DC \\ B \rightarrow bB \mid bAB \mid bb \mid aS & C \rightarrow Ca \mid bC \mid D \quad D \rightarrow bD \mid \Lambda \end{array}$$

Exercise 4.48. ♣

Show that the nullable variables defined by Definition 4.26 are precisely those variables A for which $A \Rightarrow^* \Lambda$.