- Huiswerkopgave 3: deadline 21 november 2025, 23.59 uur

Section 5

Context-Free and Non-Context-Free Languages

From lecture 2:
Regular language is language accepted by an FA.

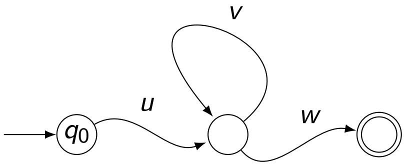
Theorem

Suppose L is a language over the alphabet Σ. If L is accepted by a finite automaton M, and if n is the number of states of M, then

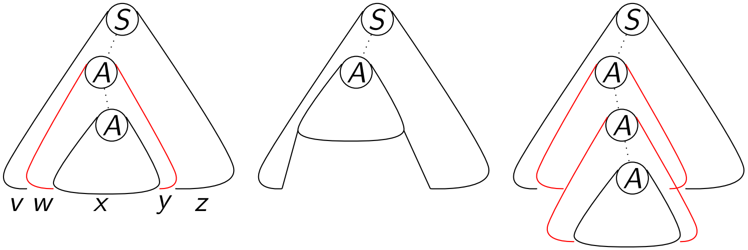
$\forall$ *for every $x \in L$*
satisfying $|x| \geq n$
$\exists$ *there are three string u, v, and w,*
such that $x = uvw$ and the following three conditions are true:
(1) $|uv| \leq n$,
(2) $|v| \geq 1$
$\forall$ *and (3) for all $i \geq 0$, uv^iw belongs to L*

[M] Thm. 2.29

From lecture 2:

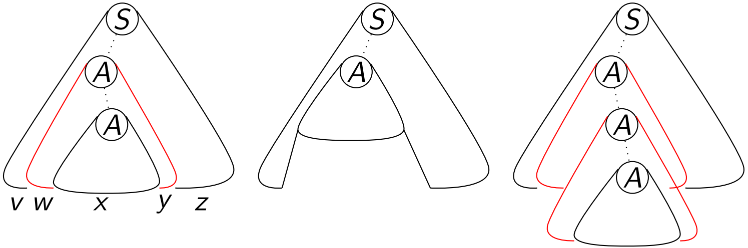


[M] Fig. 2.28


S
A
A
v w x y z
S
A
S
A
A
A

$$S \Rightarrow^* vAz \Rightarrow^* vwAyz \Rightarrow^* vwxyz, \; v, w, x, y, z \in \Sigma^*$$

$$S \underset{(1)}{\Rightarrow^*} vAz, \; A \underset{(2)}{\Rightarrow^*} wAy, \; A \underset{(3)}{\Rightarrow^*} x$$



$$S \underset{(1)}{\Rightarrow^*} vAz \underset{(3)}{\Rightarrow^*} vxz$$

$$S \underset{(1)}{\Rightarrow^*} vAz \underset{(2)}{\Rightarrow^*} vwAyz \underset{(2)}{\Rightarrow^*} vwwAyyz \underset{(3)}{\Rightarrow^*} vwwxyyz$$

Theorem (Pumping Lemma for context-free languages)

$\forall$ *for every context-free language* L

$\exists$ *there exists a constant* $n \geq 2$

such that

$\forall$ *for every* $u \in L$

with $|u| \geq n$

$\exists$ *there exists a decomposition* $u = vwxyz$

such that

(1) $|wy| \geq 1$

(2) $|wxy| \leq n$,

$\forall$ *(3) for all* $m \geq 0$, $vw^m xy^m z \in L$

[M] Thm. 6.1

Example

AnBnCn is not context-free.

[M] E 6.3

Application of pumping lemma:
mainly to prove that a language L cannot be generated by a context-free grammar.

How?
Find a string $u \in L$ with $|u| \geq n$ that cannot be pumped up!

What is n?
What should u be?
What can v, w, x, y and z be?
What should m be?

Suppose that there exists context-free grammar G with $L(G) = L$.
Let $n \geq 2$ be the integer from the pumping lemma.

We prove:

There exists $u \in L$ with $|u| \geq n$, such that
for every five strings v, w, x, y and z such that $u = vwxyz$
if

1. $|wy| \geq 1$
2. $|wxy| \leq n$

then

3. there exists $m \geq 0$, such that vw^mxy^mz does not belong to L

Example

AnBnCn is not context-free.

[M] E 6.3

$u = a^n b^n c^n$

$\{ \, x \in \{a, b, c\}^* \mid n_a(x) = n_b(x) = n_c(x) \, \}$

Example

XX is not context-free.

[M] E 6.4

Example

AnBnCn is not context-free.

[M] E 6.3

$u = a^n b^n c^n$

$\{ x \in \{a, b, c\}^* \mid n_a(x) = n_b(x) = n_c(x) \}$

Example

XX is not context-free.

[M] E 6.4

$u = a^n b^n a^n b^n$

$\{ a^i b^j a^i b^j \mid i, j \geq 0 \}$

Example

$\{ x \in \{a, b, c\}^* \mid n_a(x) < n_b(x)$ and $n_a(x) < n_c(x) \}$ is not context-free.

[M] E 6.5

ABOVE

$L = \{ x \in \{a, b, c\}^* \mid n_a(x) < n_b(x)$ and $n_a(x) < n_c(x) \}$ is not context-free.

Proof by contradiction.

Suppose L is context-free, then there exists a pumping constant n for L.

Choose $u = a^n b^{n+1} c^{n+1}$. Then $u \in L$, and $|u| \geq n$.

This means that we can pump u within the language L.

Consider a decomposition $u = vwxyz$ that satisfies the pumping lemma, in particular $|wxy| \leq n$.

Case 1: wy contains a letter a. Then wy cannot contain letter c (otherwise $|wxy| > n$). Now $u_2 = vw^2xy^2z$ contains more a's than u, so at least $n + 1$, while u_2 still contains $n + 1$ c's. Hence $u_2 \notin L$.

Case 2: wy contains no a. Then wy contains at least one b or one c (or both). Then $u_0 = vw^0xy^0z = vxz$ has still n a's, but less than $n + 1$ b's or less than $n + 1$ c's (depending on which letter is in wy). Hence $u_0 \notin L$.

These are the only two possibilities for the decomposition $vwxyz$, in both cases we see that pumping leads out of the language L.

Hence u cannot be pumped.

Contradiction; so L is not context-free.

Example

The Set of Legal C Programs is Not a CFL

[M] E 6.6

Choose $u =$

```
main(){int aaa...a;aaa...a=aaa...a;}
```

where `aaa...a` contains $n+1$ a's

From lecture 2:

FA $M_i = (Q_i, \Sigma, q_i, A_i, \delta_i) \quad i = 1, 2$

Product construction

construct FA $M = (Q, \Sigma, q_0, A, \delta)$ such that
- $Q = Q_1 \times Q_2$
- $q_0 = (q_1, q_2)$
- $\delta(\ (p, q), \sigma) = (\ \delta_1(p, \sigma), \delta_2(q, \sigma)\ )$
- A as needed

Theorem (2.15 Parallel simulation)

- $A = \{(p, q) \mid p \in A_1$ *or* $q \in A_2\}$, *then* $L(M) = L(M_1) \cup L(M_2)$
- $A = \{(p, q) \mid p \in A_1$ *and* $q \in A_2\}$, *then* $L(M) = L(M_1) \cap L(M_2)$
- $A = \{(p, q) \mid p \in A_1$ *and* $q \notin A_2\}$, *then* $L(M) = L(M_1) - L(M_2)$

Proof. . .

[M] Sect 2.2

From lecture 5:

Regular languages are closed under

- Boolean operations (complement, union, intersection, minus)
- Regular operations (union, concatenation, star)
- Reverse (mirror)
- (inverse) Homomorphism (not treated this year)

From lecture 6:
Using building blocks

Theorem

If L_1, L_2 are CFL, then so are $L_1 \cup L_2$, $L_1 L_2$ and L_1^*.

$G_i = (V_i, \Sigma, S_i, P_i)$, having no variables in common.

Construction

$G = (V_1 \cup V_2 \cup \{S\}, \Sigma, S, P)$, new axiom S
– $P = P_1 \cup P_2 \cup \{S \to S_1, S \to S_2\}$ $L(G) = L(G_1) \cup L(G_2)$
– $P = P_1 \cup P_2 \cup \{S \to S_1 S_2\}$ $L(G) = L(G_1)L(G_2)$

$G = (V_1 \cup \{S\}, \Sigma, S, P)$, new axiom S
– $P = P_1 \cup \{S \to SS_1, S \to \Lambda\}$ $L(G) = L(G_1)^*$

[M] Thm 4.9

How about

- $L_1 \cap L_2$
- $L_1 - L_2$
- L_1'

for CFLs L_1 and L_2 ?

From lecture 6:

Example

AnBnCn is intersection of two context-free languages.

$L_1 = \{a^i b^i c^k \mid i, k \geq 0\}$
$L_2 = \{a^i b^k c^k \mid i, k \geq 0\}$
[M] E 6.10

Hence, CFL is not closed under intersection

Example

AnBnCn is intersection of two context-free languages.

[M] E 6.10

Hence, CFL is not closed under intersection

$L_1 \cap L_2 = (L_1' \cup L_2')'$

Hence, CFL is not closed under complement

$L_1' = \Sigma^* - L_1$

Hence, CFL is not closed under setminus

Example

Complement of XX

$= \{ x \in \{a, b\}^* \mid |x| \text{ is odd } \} \cup \{ x\,y \mid x, y \in \{a, b\}^*, |x| = |y|, x \neq y \}$

is context-free

[M] E 6.11

Indeed, CFL is not closed under complement

Example

Complement of *AnBnCn* is context-free.

[M] E 6.12

Example

Complement of *AnBnCn* is context-free.

$AnBnCn = L_1 \cap L_2 \cap L_3$, with
$L_1 = \{a^i b^j c^k \mid i \leq j\}$
$L_2 = \{a^i b^j c^k \mid j \leq k\}$
$L_3 = \{a^i b^j c^k \mid k \leq i\}$
[M] E 6.12

Example

Complement of $\{x \in \{a, b, c\}^* \mid n_a(x) = n_b(x) = n_c(x)\}$ is context-free.

$\{x \in \{a, b, c\}^* \mid n_a(x) = n_b(x) = n_c(x)\} = A_1 \cap A_2 \cap A_3$, with
$A_1 = \{x \in \{a, b, c\}^* \mid n_a(x) \leq n_b(x)\}$
$A_2 = \{x \in \{a, b, c\}^* \mid n_b(x) \leq n_c(x)\}$
$A_3 = \{x \in \{a, b, c\}^* \mid n_c(x) \leq n_a(x)\}$
[M] E 6.12

Example

$L_1 = \{\ a^{2n} b^n \mid n \geq 1\ \}^*$

$a^{16} b^8 a^8 b^4 a^4 b^2 a^2 b^1$

$L_2 = a^* \{\ b^n a^n \mid n \geq 1\ \}^* \{\ b\ \}$

"given a CFL L, does it have property ... ?" yes/no
input CFG G

Given CFG G $[G_1$ and $G_2]$
– and given a string x, is $x \in L(G)$? membership problem
try all derivations up to...
Cocke, Younger, and Kasami (1967): n^3 (with DP)
Earley (1970): n^3 (and n^2 if G is unambiguous)

– is $L(G_1) \cap L(G_2)$ nonempty? [M] Thm 9.20
– is $L(G) = \Sigma^*$? [M] Thm 9.23
– is $L(G_1) \subseteq L(G_2)$?

$L(G) = \Sigma^*$, if and only if $\Sigma^* \subseteq L(G)$

All undecidable

Thanks to HJH for the slides so far