Exercise 9.1.

Show that the relation $\leq$ on the set of decision problems is reflexive and transitive.

Give an example to show that it is not symmetric.

Exercise 9.5.

Fermat's last theorem, until recently one of the most famous unproved statements in mathematics, asserts that there are no integer solutions (x, y, z, n) to the equation $x^n + y^n = z^n$ satisfying $x, y > 0$ and $n > 2$.

Ignoring the fact that the theorem has now been proved, explain how a solution to the halting problem would allow you to determine the truth or falsity of the statement.

A slide from lecture 7:

Accepts: Given a TM T and a string x, is $x \in L(T)$?
Instances are . . .

Halts: Given a TM T and a string x, does T halt on input x ?
Instances are . . .

Self-Accepting: Given a TM T, does T accept the string $e(T)$?
Instances are . . .

Now fix a TM T:
T-*Accepts*: Given a string x, does T accept x ?
Instances are . . .
Decidable or undecidable ? (cf. **Exercise 9.7.**)

Exercise 9.7.

As discussed at the beginning of Section 9.3, there is at least one TM T such that the decision problem

"Given w, does T accept w ?"

is unsolvable.

Show that every TM accepting a nonrecursive language has this property.

A slide from lecture 7:

Reduction from *Accepts* to *Accepts*-Λ.

Instance of *Accepts* is (T_1, x) for TM T_1 and string x.
Instance of *Accepts*-Λ is TM T_2.

$T_2 = F(T_1, x) =$

$$\textit{Write}(x) \rightarrow T_1$$

T_2 accepts Λ, if and only if T_1 accepts x.

Exercise 9.8.

Show that for every $w \in \Sigma^*$, the problem *Accepts* can be reduced to the problem:

Given a TM T, does T accept w?

(This shows that, just as *Accepts*-Λ is unsolvable, so is *Accepts*$-w$, for every w.)

Accepts-Λ: Given a TM T, is $\Lambda \in L(T)$?

Exercise 9.9.

Construct a reduction from *Accepts*-Λ to *Accepts*-$\{\Lambda\}$:

Given a TM T, is $L(T) = \{\Lambda\}$?

A slide from lecture 7

Theorem 9.9. The following five decision problems are undecidable.

4. *Equivalent*: Given two TMs T_1 and T_2, is $L(T_1) = L(T_2)$

Proof.

4. Prove that *Subset* $\leq$ *Equivalent* . . .

Subset: Given two TMs T_1 and T_2, is $L(T_1) \subseteq L(T_2)$?

Equivalent: Given two TMs T_1 and T_2, is $L(T_1) = L(T_2)$

Exercise 9.10.

a. Given two sets A and B, find two sets C and D, defined in terms of A and B, such that $A = B$ if and only if $C \subseteq D$.

b. Show that the problem *Equivalent* can be reduced to the problem *Subset*.

AcceptsEverything:

Given a TM T with input alphabet Σ, is $L(T) = \Sigma^*$?

Equivalent: Given two TMs T_1 and T_2, is $L(T_1) = L(T_2)$

Exercise 9.11. Construct a reduction from *AcceptsEverything* to the problem *Equivalent*.

Exercise 9.23. Show that the property "accepts its own encoding" is not a language property of TMs.

Part of a slide from lecture 4:

Definition 7.33. An Encoding Function (continued)

For each move m of T of the form $\delta(p, \sigma) = (q, \tau, D)$

$$e(m) = 1^{n(p)}01^{n(\sigma)}01^{n(q)}01^{n(\tau)}01^{n(D)}0$$

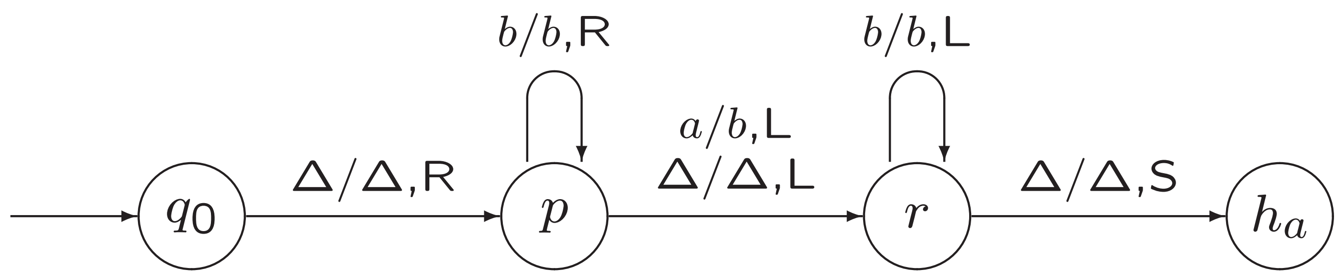
We list the moves of T in some order as $m_1, m_2, \ldots, m_k$, and we define

$$e(T) = e(m_1)0e(m_2)0\ldots0e(m_k)0$$

Exercise 9.23. Show that the property "accepts its own encoding" is not a language property of TMs.

A slide from lecture 4:

Example 7.34. A Sample Encoding of a TM



<u>1</u>11010111101010 0 11110111011110111010 0
111101101111101110110 0 111101011111010110 0
11111011101111101110110 0 1111101010101110 0

Exercise 9.12.
For each decision problem below, determine whether it is decidable or undecidable, and prove your answer.

a. Given a TM T, does it ever reach a nonhalting state other than its initial state if it starts with a blank tape?

Exercise 9.12.
For each decision problem below, determine whether it is decidable or undecidable, and prove your answer.

b. Given a TM T and a nonhalting state q of T, does T ever enter state q when it begins with a blank tape?

e. Given a TM T, is there a string it accepts in an even number of moves?

j. Given a TM T, does T halt within ten moves on every string?

l. Given a TM T, does T eventually enter every one of its nonhalting states if it begins with a blank tape?

Exercise 9.13.

In this problem TMs are assumed to have input alphabet $\{0, 1\}$. For a finite set $S \subseteq \{0, 1\}^*$, P_S denotes the decision problem: Given a TM T, is $S \subseteq L(T)$?

a. Show that if $x, y \in \{0, 1\}^*$, then $P_{\{x\}} \leq P_{\{y\}}$.

b. Show that if $x, y, z \in \{0, 1\}^*$, then $P_{\{x\}} \leq P_{\{y,z\}}$.

c. Show that if $x, y, z \in \{0, 1\}^*$, then $P_{\{x,y\}} \leq P_{\{z\}}$.

d. Is it true that for every two finite subsets S and U of $\{0, 1\}^*$, $P_S \leq P_U$.