Computability

voorjaar 2024

https://liacs.leidenuniv.nl/~vlietrvan1/computability/

college 6, 15 maart 2024

8.3. More General Grammars
9. Undecidable Problems
9.1. A Language That Can't Be Accepted,
and a Problem That Can't Be Decided
9.2. Reductions and the Halting Problem
9.3. More Decision Problems Involving Turing Machines

Huiswerkopgave

inleverdatum voor 0.2 punt: 22 maart 2024, 23.59 uur

doel bij nakijken eerste inzendingen: vóór 28 maart

Theorem 8.14.
For every Turing machine T with input alphabet Σ,
there is an unrestricted grammar G
generating the language $L(T) \subseteq \Sigma^*$.

Proof.
1. Generate (every possible) input string for T.
2. Simulate computation of T for this input string as derivation in grammar.
3. If T reaches accept state, reconstruct original input string.

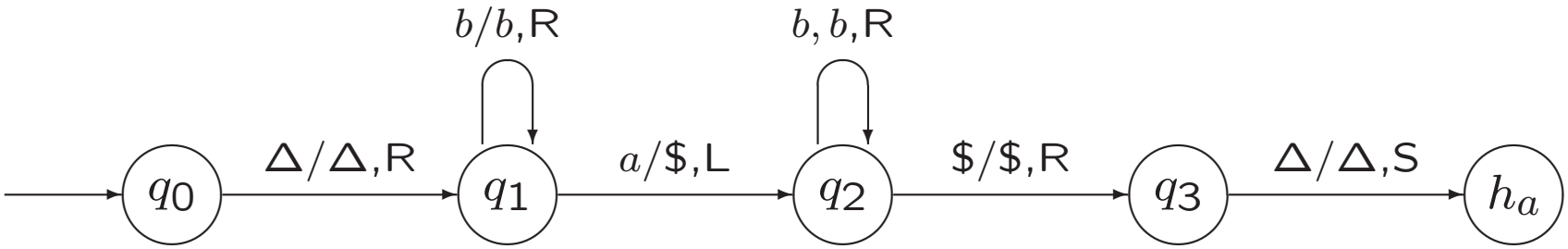
(Part of) a slide from lecture 2

Notation:

description of tape contents: $x\underline{\sigma}y$ or $x\underline{y}$

configuration $xqy = xqy\Delta = xqy\Delta\Delta$

initial configuration corresponding to input x: $q_0\Delta x$

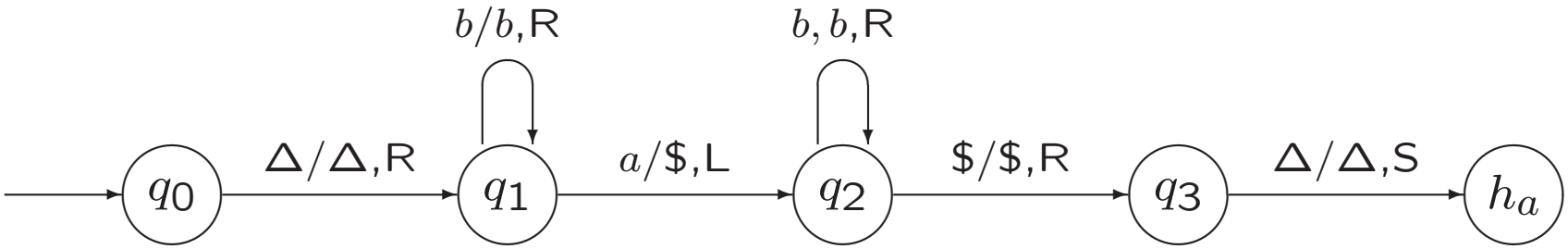


Computation for $x = ba$:

$$q_0\Delta ba\Delta \vdash \Delta q_1 ba\Delta \vdash \Delta b q_1 a\Delta \vdash \Delta q_2 b\$\Delta \vdash \Delta b q_2 \$\Delta \vdash \Delta b\$ q_3\Delta \vdash \Delta b\$ h_a\Delta \vdash$$

Theorem 8.14.
For every Turing machine T with input alphabet Σ,
there is an unrestricted grammar G
generating the language $L(T) \subseteq \Sigma^*$.

Proof.
1. Generate (every possible) input string for T (two copies),
with additional ($\Delta\Delta$)'s and state.
2. Simulate computation of T for this input string as derivation
in grammar (on second copy).
3. If T reaches accept state, reconstruct original input string.



Computation for $x = ba$:

$$q_0\Delta ba\Delta \vdash \Delta q_1 ba\Delta \vdash \Delta b q_1 a\Delta \vdash \Delta q_2 b\$\Delta \vdash \Delta b q_2 \$\Delta \vdash \Delta b\$ q_3\Delta \vdash \Delta b\$ h_a\Delta \vdash$$

Theorem 8.14.
For every Turing machine T with input alphabet Σ,
there is an unrestricted grammar G
generating the language $L(T) \subseteq \Sigma^*$.

Proof.
1. Generate (every possible) input string for T (two copies),
with additional ($\Delta\Delta$)'s and state.
2. Simulate computation of T for this input string as derivation
in grammar (on second copy).
3. If T reaches accept state, reconstruct original input string.

Ad 2. Move $\delta(p, a) = (q, b, R)$ of T
yields production $p(\sigma_1 a) \to (\sigma_1 b)q$
Move $\delta(p, a) = (q, b, L)$ of T
yields production $(\sigma_1\sigma_2)p(\sigma_3 a) \to q(\sigma_1\sigma_2)(\sigma_3 b)$
Move $\delta(p, a) = (q, b, S)$ of T
yields production $p(\sigma_1 a) \to q(\sigma_1 b)$

3. If T reaches accept state, reconstruct original input string. . .

Theorem 8.14.
For every Turing machine T with input alphabet Σ,
there is an unrestricted grammar G
generating the language $L(T) \subseteq \Sigma^*$.

Proof.
1. Generate (every possible) input string for T (two copies),
with additional ($\Delta\Delta$)'s and state.
2. Simulate computation of T for this input string as derivation
in grammar (on second copy).
3. If T reaches accept state, reconstruct original input string.

Ad 3. Propagate h_a all over the string
$h_a(\sigma_1\sigma_2) \to \sigma_1$, for $\sigma_1 \in \Sigma$
$h_a(\Delta\sigma_2) \to \Lambda$

Theorem 8.14.
For every Turing machine T with input alphabet Σ,
there is an unrestricted grammar G
generating the language $L(T) \subseteq \Sigma^*$.

Proof.
1. Generate (every possible) input string for T (two copies),
with additional ($\Delta\Delta$)'s and state.
2. Simulate computation of T for this input string as derivation
in grammar (on second copy).
3. If T reaches accept state, reconstruct original input string.

Ad 3. Propagate h_a all over the string (too few / many h_a's. . .)
$h_a(\sigma_1\sigma_2) \to \sigma_1$, for $\sigma_1 \in \Sigma$
$h_a(\Delta\sigma_2) \to \Lambda$

8.5. Not Every Language is Recursively Enumerable

reg. languages	FA	reg. grammar	reg. expression
determ. cf. languages	DPDA		
cf. languages	PDA	cf. grammar	
cs. languages	LBA	cs. grammar	
re. languages	TM	unrestr. grammar	

9. Undecidable Problems

9.1. A Language That Can't Be Accepted, and a Problem That Can't Be Decided

A slide from lecture 4

Definition 8.1. Accepting a Language and Deciding a Language

A Turing machine T with input alphabet Σ accepts a language $L \subseteq \Sigma^*$,
if $L(T) = L$.

T *decides* L,
if T computes the characteristic function $\chi_L : \Sigma^* \to \{0, 1\}$

A language L is *recursively enumerable*,
if there is a TM that accepts L,

and L is *recursive*,
if there is a TM that decides L.

A slide from lecture 4

Definition 7.33. An Encoding Function

Assign numbers to each state:
$n(h_a) = 1$, $n(h_r) = 2$, $n(q_0) = 3$, $n(q) \geq 4$ for other $q \in Q$.

Assign numbers to each tape symbol:
$n(a_i) = i$.

Assign numbers to each tape head direction:
$n(R) = 1$, $n(L) = 2$, $n(S) = 3$.

A slide from lecture 4

Definition 7.33. An Encoding Function (continued)

For each move m of T of the form $\delta(p, \sigma) = (q, \tau, D)$

$$e(m) = 1^{n(p)}01^{n(\sigma)}01^{n(q)}01^{n(\tau)}01^{n(D)}0$$

We list the moves of T in some order as $m_1, m_2, \ldots, m_k$, and we define

$$e(T) = e(m_1)0e(m_2)0 \ldots 0e(m_k)0$$

If $z = z_1 z_2 \ldots z_j$ is a string, where each $z_i \in \mathcal{S}$,

$$e(z) = 01^{n(z_1)}01^{n(z_2)}0 \ldots 01^{n(z_j)}0$$

	$e(T_0)$	$e(T_1)$	$e(T_2)$	$e(T_3)$	$e(T_4)$	$e(T_5)$	$e(T_6)$	$e(T_7)$	$e(T_8)$	$e(T_9)$
$L(T_0)$	1	0	1	0	0	1	0	0	0	1
$L(T_1)$	0	1	1	1	0	0	0	0	1	0
$L(T_2)$	1	0	0	1	0	0	1	0	0	0
$L(T_3)$	0	0	0	0	0	0	0	0	0	0
$L(T_4)$	0	0	0	0	1	0	0	0	0	0
$L(T_5)$	0	0	1	1	0	1	0	1	0	0
$L(T_6)$	0	0	0	0	0	0	0	0	1	0
$L(T_7)$	1	1	1	1	1	1	1	1	1	1
$L(T_8)$	0	1	0	1	0	1	0	1	0	1
$L(T_9)$	0	0	0	0	0	0	0	0	0	0
$\ldots$						$\ldots$				

	$e(T_0)$	$e(T_1)$	$e(T_2)$	$e(T_3)$	$e(T_4)$	$e(T_5)$	$e(T_6)$	$e(T_7)$	$e(T_8)$	$e(T_9)$
$L(T_0)$	1	0	1	0	0	1	0	0	0	1
$L(T_1)$	0	1	1	1	0	0	0	0	1	0
$L(T_2)$	1	0	0	1	0	0	1	0	0	0
$L(T_3)$	0	0	0	0	0	0	0	0	0	0
$L(T_4)$	0	0	0	0	1	0	0	0	0	0
$L(T_5)$	0	0	1	1	0	1	0	1	0	0
$L(T_6)$	0	0	0	0	0	0	0	0	1	0
$L(T_7)$	1	1	1	1	1	1	1	1	1	1
$L(T_8)$	0	1	0	1	0	1	0	1	0	1
$L(T_9)$	0	0	0	0	0	0	0	0	0	0
...						...				
NSA	0	0	1	1	0	0	1	0	1	1

Hence, *NSA* is not recursively enumerable.

A slide from lecture 4

Some Crucial features of any encoding function e:

1. It should be possible to decide algorithmically, for any string $w \in \{0,1\}^*$, whether w is a legitimate value of e.
2. A string w should represent at most one Turing machine with a given input alphabet Σ, or at most one string z.
3. If $w = e(T)$ or $w = e(z)$, there should be an algorithm for *decoding* w.

Set-up of constructing language *NSA* that is not RE:

1. Start with list of RE languages over $\{0,1\}$
   (which are subsets of $\{0,1\}^*$): $L(T_0), L(T_1), L(T_2), \ldots$
   each one associated with specific element of $\{0,1\}^*$
   (namely $e(T_i)$)

2. Define another language *NSA* by:
   $e(T_i) \in NSA \iff e(T_i) \notin L(T_i)$

3. Conclusion: for all i, $NSA \neq L(T_i)$
   Hence, *NSA* is not RE

Set-up of constructing language *NSA* that is not RE:

1. Start with collection of RE languages over $\{0,1\}$
   (which are subsets of $\{0,1\}^*$): $\{L(T) \mid \text{ TM } T\}$
   each one associated with specific element of $\{0,1\}^*$
   (namely $e(T)$)

2. Define another language *NSA* by:
   $e(T) \in \mathit{NSA} \iff e(T) \notin L(T)$

3. Conclusion: for all TM T, $\mathit{NSA} \neq L(T)$
   Hence, *NSA* is not RE

Set-up of constructing language that is not RE:

1. Start with list of RE languages over $\{0,1\}$
   (which are subsets of $\{0,1\}^*$): $L(T_0), L(T_1), L(T_2), \ldots$
   each one associated with specific element of $\{0,1\}^*$

2. Define another language L by:
   $x \in L \iff x \notin$ (language that x is associated with)

3. Conclusion: for all i, $L \neq L(T_i)$
   Hence, L is not RE

Set-up of constructing language L that is not RE:

1. Start with list of RE languages over $\{0,1\}$
   (which are subsets of $\{0,1\}^*$): $L(T_0), L(T_1), L(T_2), \ldots$
   each one associated with specific element of $\{0,1\}^*$
   (namely x_i)

2. Define another language L by:
   $x_i \in L \iff x_i \notin L(T_i)$

3. Conclusion: for all i, $L \neq L(T_i)$
   Hence, L is not RE

Every infinite list $x_0, x_1, x_2, \ldots$ of different elements of $\{0,1\}^*$ yields language L that is not RE

	Λ	0	1	00	01	10	11	000	001	010	...
$L(T_0)$	1	0	1	0	0	1	0	0	0	1	...
$L(T_1)$	0	1	1	1	0	0	0	0	1	0	...
$L(T_2)$	1	0	0	1	0	0	1	0	0	0	...
$L(T_3)$	0	0	0	0	0	0	0	0	0	0	...
$L(T_4)$	0	0	0	0	1	0	0	0	0	0	...
$L(T_5)$	0	0	1	1	0	1	0	1	0	0	...
$L(T_6)$	0	0	0	0	0	0	0	0	1	0	...
$L(T_7)$	1	1	1	1	1	1	1	1	1	1	...
$L(T_8)$	0	1	0	1	0	1	0	1	0	1	...
$L(T_9)$	0	0	0	0	0	0	0	0	0	0	...
...							...				
newL	0	0	1	1	0	0	1	0	1	1	...

Hence, newL is not recursively enumerable.

Definition 9.1. The Languages *NSA* and *SA*

Let

$$\begin{aligned} NSA &= \{e(T) \mid T \text{ is a TM, and } e(T) \notin L(T)\} \\ SA &= \{e(T) \mid T \text{ is a TM, and } e(T) \in L(T)\} \end{aligned}$$

(*NSA* and *SA* are for "non-self-accepting" and "self-accepting.")

A slide from lecture 4

Some Crucial features of any encoding function e:

1. It should be possible to decide algorithmically, for any string $w \in \{0,1\}^*$, whether w is a legitimate value of e.
2. A string w should represent at most one Turing machine with a given input alphabet Σ, or at most one string z.
3. If $w = e(T)$ or $w = e(z)$, there should be an algorithm for *decoding* w.

Theorem 9.2. The language NSA is not recursively enumerable. The language SA is recursively enumerable but not recursive.

Proof. . .

Exercise 9.2.

Describe how a universal Turing machine could be used in the proof that SA is recursively enumerable.

Given a TM T, does T accept the string $e(T)$?

Decision problem: problem for which the answer is 'yes' or 'no':

Given . . . , is it true that . . . ?

Given an undirected graph $G = (V, E)$,
does G contain a Hamiltonian path?

Given a list of integers $x_1, x_2, \ldots, x_n$,
is the list sorted?

Self-Accepting: Given a TM T, does T accept the string $e(T)$?

instances. . .

Decision problem: problem for which the answer is ‘yes’ or ‘no’:

Given . . . , is it true that . . . ?

yes-instances of a decision problem:
instances for which the answer is ‘yes’

no-instances of a decision problem:
instances for which the answer is ‘no’

Self-Accepting: Given a TM T, does T accept the string $e(T)$?

Three languages corresponding to this problem:

1. *SA*: strings representing yes-instances
2. *NSA*: strings representing no-instances
3. . . .

Self-Accepting: Given a TM T, does T accept the string $e(T)$?

Three languages corresponding to this problem:

1. *SA*: strings representing yes-instances
2. *NSA*: strings representing no-instances
3. E': strings not representing instances

For general decision problem P,
an encoding e of instances I as strings $e(I)$ over alphabet Σ
is called *reasonable*, if

1. there is algorithm to decide if string over Σ is encoding $e(I)$
2. e is injective
3. string $e(I)$ can be decoded

A slide from lecture 4

Some Crucial features of any encoding function e:

1. It should be possible to decide algorithmically, for any string $w \in \{0,1\}^*$, whether w is a legitimate value of e.
2. A string w should represent at most one Turing machine with a given input alphabet Σ, or at most one string z.
3. If $w = e(T)$ or $w = e(z)$, there should be an algorithm for *decoding* w.

For general decision problem P and reasonable encoding e,

$$\begin{aligned} Y(P) &= \{e(I) \mid I \text{ is yes-instance of } P\} \\ N(P) &= \{e(I) \mid I \text{ is no-instance of } P\} \\ E(P) &= Y(P) \cup N(P) \end{aligned}$$

$E(P)$ must be recursive

Definition 9.3. Decidable Problems

If P is a decision problem, and e is a reasonable encoding of instances of P over the alphabet Σ, we say that P is *decidable* if $Y(P) = \{e(I) \mid \ I$ is a yes-instance of $P\}$ is a recursive language.

Theorem 9.4. The decision problem *Self-Accepting* is undecidable.

Proof. . .

For every decision problem, there is *complementary* problem P', obtained by changing 'true' to 'false' in statement.

Non-Self-Accepting:
Given a TM T, does T fail to accept $e(T)$?

Theorem 9.5. For every decision problem P, P is decidable if and only if the complementary problem P' is decidable.

Proof. . .

SA vs. *NSA*

Self-Accepting vs. *Non-Self-Accepting*

9.2. Reductions and the Halting Problem

(Informal) Examples of reductions

1. Recursive algorithms

2. Given NFA M and string x, is $x \in L(M)$?

3. Given FAs M_1 and M_2, is $L(M_1) \subseteq L(M_2)$?

Theorem 2.15.
Suppose $M_1 = (Q_1, \Sigma, q_1, A_1, \delta_1)$ and $M_2 = (Q_2, \Sigma, q_2, A_2, \delta_2)$ are finite automata accepting L_1 and L_2, respectively.
Let M be the FA $(Q, \Sigma, q_0, A, \delta)$, where

$Q = Q_1 \times Q_2$

$q_0 = (q_1, q_2)$

and the transition function δ is defined by the formula

$\delta((p, q), \sigma) = (\delta_1(p, \sigma), \delta_2(q, \sigma))$

for every $p \in Q_1$, every $q \in Q_2$, and every $\sigma \in \Sigma$.

Then

1. If $A = \{(p, q) |\ p \in A_1$ or $q \in A_2\}$,
   M accepts the language $L_1 \cup L_2$.
2. If $A = \{(p, q) |\ p \in A_1$ and $q \in A_2\}$,
   M accepts the language $L_1 \cap L_2$.
3. If $A = \{(p, q) |\ p \in A_1$ and $q \notin A_2\}$,
   M accepts the language $L_1 - L_2$.

(Informal) Examples of reductions

3. *SubsetFA*: Given FAs M_1 and M_2, is $L(M_1) \subseteq L(M_2)$?

3'. *AcceptsNothingFA*: Given FA M, is $L(M) = \emptyset$?

Definition 9.6. Reducing One Decision Problem to Another . . .

Suppose P_1 and P_2 are decision problems. We say P_1 is reducible to P_2 ($P_1 \leq P_2$)

- if there is an algorithm
- that finds, for an arbitrary instance I of P_1, an instance $F(I)$ of P_2,
- such that

  for every I the answers for the two instances are the same,

  or I is a yes-instance of P_1

  if and only if $F(I)$ is a yes-instance of P_2.

. . .

Theorem 9.7.

. . .

Suppose P_1 and P_2 are decision problems, and $P_1 \leq P_2$. If P_2 is decidable, then P_1 is decidable.

Informal proof:
Suppose that $P_1 \leq P_2$, and that function F maps instance I_1 of P_1 to instance $I_2 = F(I_1)$ of P_2 with same answer yes/no

If we have an algorithm/TM A_2 to solve P_2,
then we also have an algorithm/TM A_1 to solve P_1,
as follows:

A_1:
Given instance I_1 of P_1,
1. construct $I_2 = F(I_1)$;
2. run A_2 on I_2.

$$A_1 : \quad I_1 \xrightarrow[F]{} I_2 \xrightarrow[A_2]{} \text{yes/no}$$

A_1 answers 'yes' for I_1,
if and only if A_2 answers 'yes' for I_2,
if and only $I_2 = F(I_1)$ is yes-instance of P_2,
if and only if I_1 is yes-instance of P_1

Two more decision problems:

Accepts: Given a TM T and a string w, is $w \in L(T)$?

Halts: Given a TM T and a string w, does T halt on input w ?

Theorem 9.8. Both *Accepts* and *Halts* are undecidable.

Proof.

1. Prove that *Self-Accepting* $\leq$ *Accepts* . . .

Definition 9.6. Reducing One Decision Problem to Another . . .

Suppose P_1 and P_2 are decision problems. We say P_1 is reducible to P_2 ($P_1 \leq P_2$)

- if there is an algorithm
- that finds, for an arbitrary instance I of P_1, an instance $F(I)$ of P_2,
- such that

  for every I the answers for the two instances are the same,

  or I is a yes-instance of P_1

  if and only if $F(I)$ is a yes-instance of P_2.

. . .

Theorem 9.8. Both *Accepts* and *Halts* are undecidable.

Proof.

1. Prove that *Self-Accepting* $\leq$ *Accepts* . . .

2. Prove that *Accepts* $\leq$ *Halts* . . .

Application:

```
n = 4;
while (n is the sum of two primes)
  n = n+2;
```

This program loops forever, if and only if Goldbach's conjecture is true.