Computability

voorjaar 2024

https://liacs.leidenuniv.nl/~vlietrvan1/computability/

college 5, 8 maart 2024

8. Recursively Enumerable Languages
8.3. More General Grammars

Huiswerkopgave

voor 0.4pt

individueel

inleveren: uiterlijk vanavond, 23.59 uur

A slide from lecture 4

Definition 8.1. Accepting a Language and Deciding a Language

A Turing machine T with input alphabet Σ accepts a language $L \subseteq \Sigma^*$,
if $L(T) = L$.

T decides L,
if T computes the characteristic function $\chi_L : \Sigma^* \to \{0, 1\}$

A language L is *recursively enumerable*,
if there is a TM that accepts L,

and L is *recursive*,
if there is a TM that decides L.

A slide from lecture 4

Theorem 8.2.
Every recursive language is recursively enumerable.

Proof. . .

A slide from lecture 4

Theorem 8.4. If L_1 and L_2 are both recursively enumerable languages over Σ, then $L_1 \cup L_2$ and $L_1 \cap L_2$ are also recursively enumerable.

Proof. . .

Theorem 8.5. If L_1 and L_2 are both recursive languages over Σ, then $L_1 \cup L_2$ and $L_1 \cap L_2$ are also recursive.

Proof. Exercise 8.1.

Exercise 8.1.

Show that if L_1 and L_2 are recursive languages, then $L_1 \cup L_2$ and $L_1 \cap L_2$ are also.

Theorem 8.6. If L is a recursive language over Σ, then its complement L' is also recursive.

Proof. . .

Theorem 8.7. If L is a recursively enumerable language, and its complement L' is also recursively enumerable, then L is recursive
(and therefore, by Theorem 8.6, L' is recursive).

Proof. . .

Corollary.
Let L be a recursively enumerable language.
Then

L' is recursively enumerable,
if and only
if L is recursive.

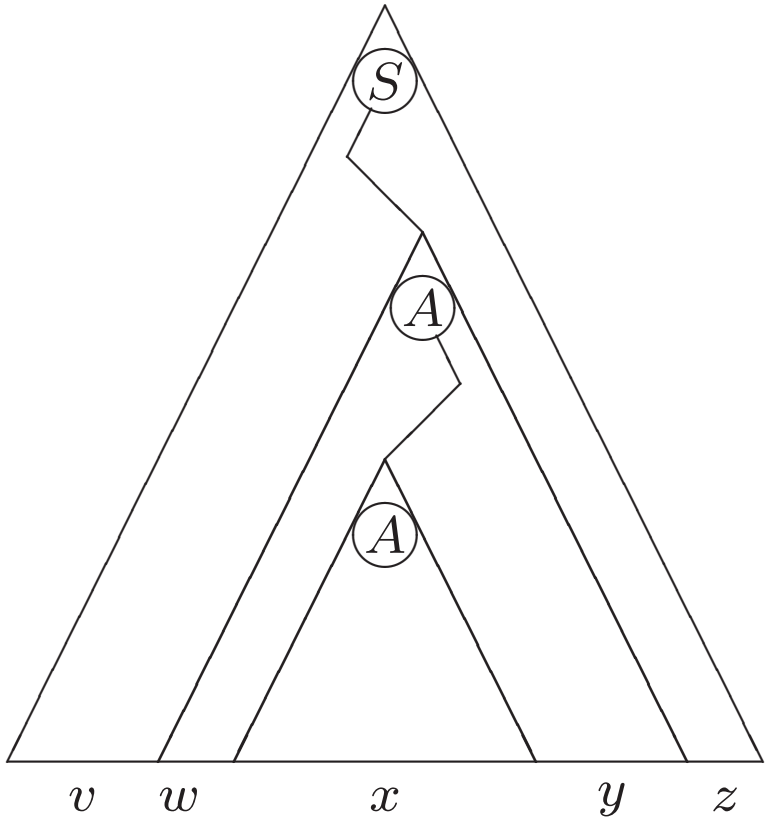
Corollary.
There exist languages that are not recursively enumerable,
if and only if
there exist languages that are not recursive.

8.3. More General Grammars

reg. languages	FA	reg. grammar	reg. expression
determ. cf. languages	DPDA		
cf. languages	PDA	cf. grammar	
cs. languages	LBA	cs. grammar	
re. languages	TM	unrestr. grammar	

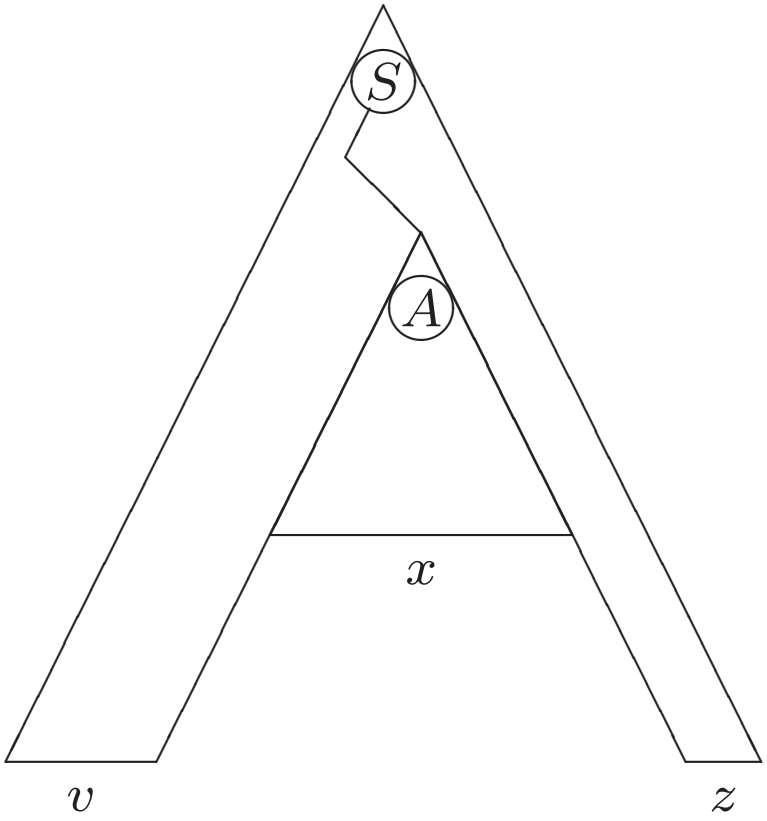
A slide from lecture 1

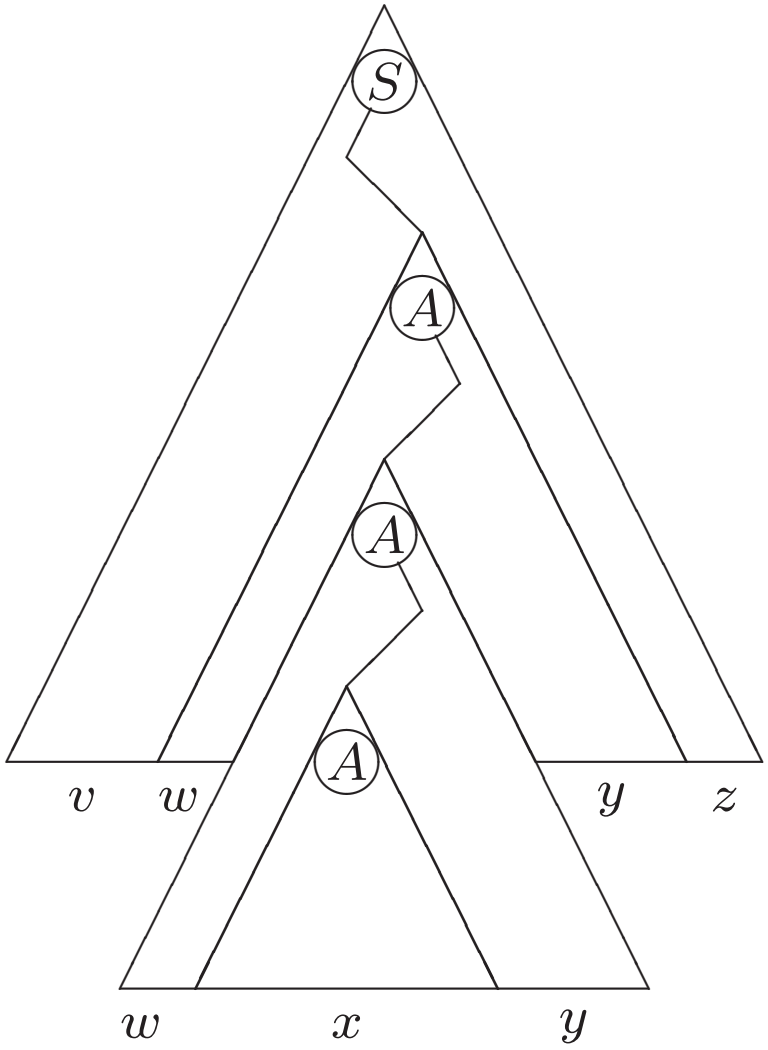
FI2: Pumping Lemma for CFLs



A slide from lecture 1

FI2: Pumping Lemma for CFLs





Definition 8.10. Unrestricted grammars

An unrestricted grammar is a 4-tuple $G = (V, \Sigma, S, P)$, where V and Σ are disjoint finite sets of variables and terminals, respectively, S is an element of V called the start symbol, and P is a finite set of productions of the form

$$\alpha \to \beta$$

where $\alpha, \beta \in (V \cup \Sigma)^*$ and α contains at least one variable.

Notation as for CFGs:

$$\alpha \Rightarrow_G^* \beta$$

$$L(G) = \{x \in \Sigma^* \mid S \Rightarrow_G^* x\}$$

but when $S \Rightarrow^* xAy \Rightarrow^* z \ \ldots$

Notation $\rightarrow$ vs $\Rightarrow$ vs $\vdash$ $\ldots$

Example 8.12. A Grammar Generating $\{a^n b^n c^n \mid n \geq 1\}$

Example 8.12. A Grammar Generating $\{a^n b^n c^n \mid n \geq 1\}$

$$S \to SABC \mid LABC$$

$$BA \to AB \quad CB \to BC \quad CA \to AC$$

$$LA \to a \quad aA \to aa \quad aB \to ab \quad bB \to bb \quad bC \to bc \quad cC \to cc$$

Example 8.11. A Grammar Generating $\{a^{2^k} \mid k \in \mathbb{N}\}$

$$\{a, a^2, a^4, a^8, a^{16}, \ldots\} = \{a, aa, aaaa, aaaaaaaa, aaaaaaaaaaaaaaaa, \ldots\}$$

Example 8.11. A Grammar Generating $\{a^{2^k} \mid k \in \mathbb{N}\}$

$$\{a, a^2, a^4, a^8, a^{16}, \ldots\} = \{a, aa, aaaa, aaaaaaaa, aaaaaaaaaaaaaaaa, \ldots\}$$

$$S \to LaR$$

$$L \to LD \qquad Da \to aaD \qquad DR \to R$$

$$L \to \Lambda \qquad R \to \Lambda$$

Example.

An Unrestricted Grammar Generating $XX = \{xx \mid x \in \{a,b\}^*\}$

First a CFG for $Pal = \{x \in \{a,b\}^* \mid x = x^r\}$:

$$S \to aSa \mid bSb \mid a \mid b \mid \Lambda$$

Example.

An Unrestricted Grammar Generating $XX = \{xx \mid x \in \{a, b\}^*\}$

$$S \to aAS \mid bBS \mid M$$

$$Aa \to aA \qquad Ab \to bA \qquad Ba \to aB \qquad Bb \to bB$$

$$AM \to Ma \qquad BM \to Mb \qquad M \to \Lambda$$

Theorem 8.13.
For every unrestricted grammar G, there is a Turing machine T with $L(T) = L(G)$.

Proof.
1. Move past input (= *NB*)
2. Simulate derivation in G on the tape of a Turing machine
3. (*PB*, followed by) Equal

Theorem 8.13.
For every unrestricted grammar G, there is a Turing machine T with $L(T) = L(G)$.

Proof.
1. Move past input
2. Simulate derivation in G on the tape of a Turing machine:
   Write S on tape
   Repeat
   a. Select production $\alpha \to \beta$
   b. Select occurrence of α (if there is one)
   c. Replace occurrence of α by β
   until b. fails (caused by . . .)
3. Equal

A slide from lecture 4

Theorem 7.31.
For every nondeterministic TM $T = (Q, \Sigma, \Gamma, q_0, \delta)$,
there is an ordinary (deterministic) TM $T_1 = (Q_1, \Sigma, \Gamma_1, q_1, \delta_1)$
with $L(T_1) = L(T)$.

Moreover, if there is no input on which T can loop forever,
then T_1 also halts on every input.

The proof of this result does not have to be known for the exam.

Example.

(The second part of) the construction from Theorem 8.13 to obtain a TM simulating a derivation in the unrestricted grammar with productions

$$S \to aBS \mid \Lambda \qquad aB \to Ba \qquad Ba \to aB \qquad B \to b$$

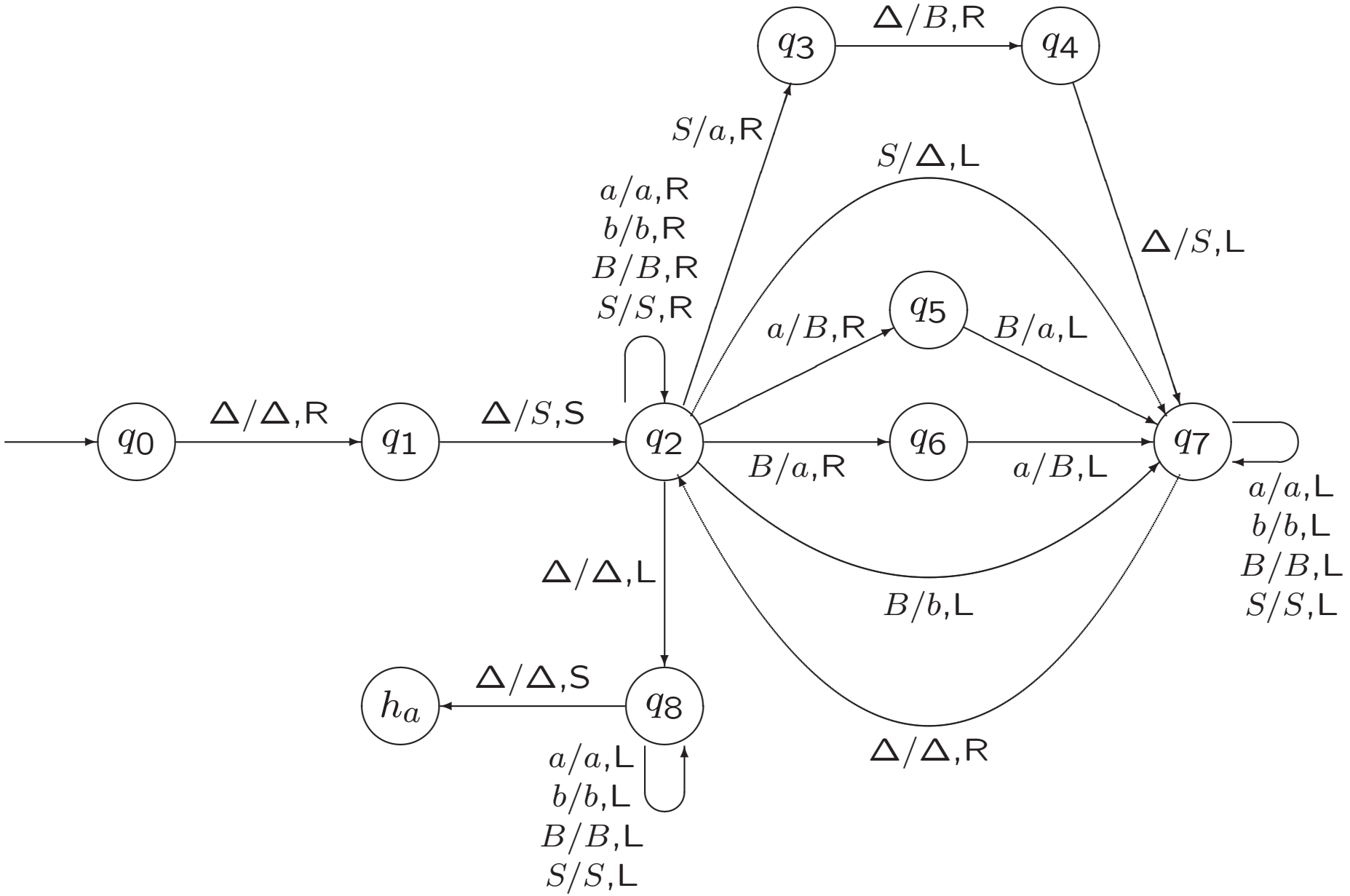
See next slide

N.B.:

In next slide, we simulate application of arbitrary production by

- first moving to arbitrary position in current string (at q_2)
- only then selecting (and applying) a possible production

This implementation of the construction must be known for the exam


q_0
q_1
q_2
q_3
q_4
q_5
q_6
q_7
q_8
h_a
Δ/Δ,R
Δ/S,S
Δ/B,R
Δ/S,L
S/a,R
S/Δ,L
a/a,R
b/b,R
B/B,R
S/S,R
a/B,R
B/a,L
B/a,R
a/B,L
a/a,L
b/b,L
B/B,L
S/S,L
Δ/Δ,L
B/b,L
Δ/Δ,S
Δ/Δ,R
a/a,L
b/b,L
B/B,L
S/S,L

Theorem 8.14.
For every Turing machine T with input alphabet Σ,
there is an unrestricted grammar G
generating the language $L(T) \subseteq \Sigma^*$.

Proof.
1. Generate (every possible) input string for T.
2. Simulate computation of T for this input string as derivation in grammar.
3. If T reaches accept state, reconstruct original input string.

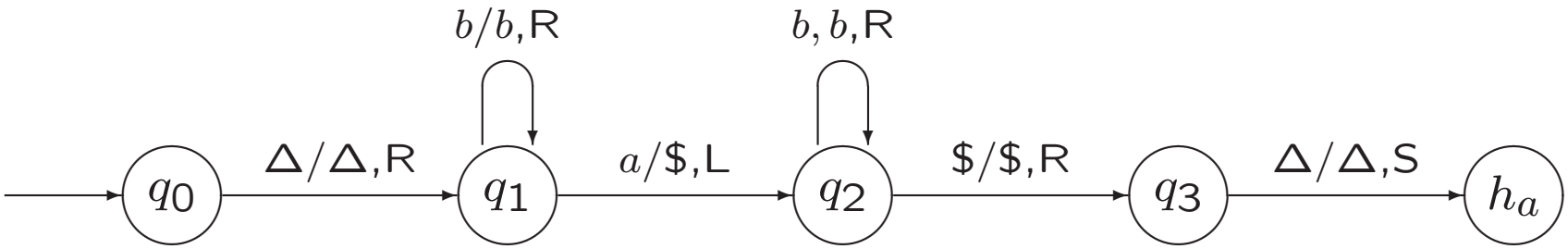
(Part of) a slide from lecture 2

Notation:

description of tape contents: $x\underline{\sigma}y$ or $x\underline{y}$

configuration $xqy = xqy\Delta = xqy\Delta\Delta$

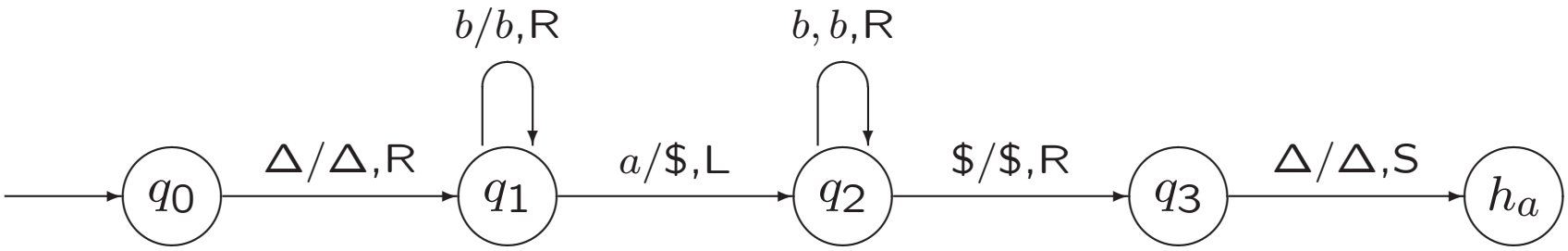
initial configuration corresponding to input x: $q_0\Delta x$



Computation for $x = ba$.

Theorem 8.14.
For every Turing machine T with input alphabet Σ,
there is an unrestricted grammar G
generating the language $L(T) \subseteq \Sigma^*$.

Proof.
1. Generate (every possible) input string for T (two copies),
with additional ($\Delta\Delta$)'s and state.
2. Simulate computation of T for this input string as derivation
in grammar (on second copy).
3. If T reaches accept state, reconstruct original input string.


b/b,R
b, b,R
q_0
Δ/Δ,R
q_1
$a/\$$,L
q_2
$\$/\$$,R
q_3
Δ/Δ,S
h_a

Theorem 8.14.
For every Turing machine T with input alphabet Σ,
there is an unrestricted grammar G
generating the language $L(T) \subseteq \Sigma^*$.

Proof.
1. Generate (every possible) input string for T (two copies),
with additional ($\triangle\triangle$)'s and state.
2. Simulate computation of T for this input string as derivation
in grammar (on second copy).
3. If T reaches accept state, reconstruct original input string.

Ad 2. Move $\delta(p, a) = (q, b, R)$ of T
yields production $p(\sigma_1 a) \to (\sigma_1 b)q$
Move $\delta(p, a) = (q, b, L)$ of T
yields production $(\sigma_1\sigma_2)p(\sigma_3 a) \to q(\sigma_1\sigma_2)(\sigma_3 b)$
Move $\delta(p, a) = (q, b, S)$ of T
yields production $p(\sigma_1 a) \to q(\sigma_1 b)$

3. If T reaches accept state, reconstruct original input string. . .

Theorem 8.14.
For every Turing machine T with input alphabet Σ,
there is an unrestricted grammar G
generating the language $L(T) \subseteq \Sigma^*$.

Proof.
1. Generate (every possible) input string for T (two copies),
with additional ($\Delta\Delta$)'s and state.
2. Simulate computation of T for this input string as derivation
in grammar (on second copy).
3. If T reaches accept state, reconstruct original input string.

Ad 3. Propagate h_a all over the string
$h_a(\sigma_1\sigma_2) \to \sigma_1$, for $\sigma_1 \in \Sigma$
$h_a(\Delta\sigma_2) \to \Lambda$

Theorem 8.14.
For every Turing machine T with input alphabet Σ,
there is an unrestricted grammar G
generating the language $L(T) \subseteq \Sigma^*$.

Proof.
1. Generate (every possible) input string for T (two copies),
with additional ($\Delta\Delta$)'s and state.
2. Simulate computation of T for this input string as derivation
in grammar (on second copy).
3. If T reaches accept state, reconstruct original input string.

Ad 3. Propagate h_a all over the string (too few / many h_a's. . .)
$h_a(\sigma_1\sigma_2) \to \sigma_1$, for $\sigma_1 \in \Sigma$
$h_a(\Delta\sigma_2) \to \Lambda$

8.5. Not Every Language is Recursively Enumerable

reg. languages	FA	reg. grammar	reg. expression
determ. cf. languages	DPDA		
cf. languages	PDA	cf. grammar	
cs. languages	LBA	cs. grammar	
re. languages	TM	unrestr. grammar	