Exercise 4.51. $\clubsuit$

A variable A in a context-free grammar $G = (V, \Sigma, S, P)$ is *live* if $A \Rightarrow^* x$ for some $x \in \Sigma^*$. A variable that is not live, is called *dead*.

Give a recursive definition, and a corresponding algorithm, for finding all live variables in G.

Exercise 4.52. ♣

A variable A in a context-free grammar $G = (V, \Sigma, S, P)$ is *reachable* if $S \Rightarrow^* \alpha A \beta$ for some $\alpha, \beta \in (\Sigma \cup V)^*$.

Give a recursive definition, and a corresponding algorithm, for finding all reachable variables in G.

Exercise 4.53.

A variable A in a context-free grammar $G = (V, \Sigma, S, P)$ is *useful* if for some string $x \in \Sigma^*$, there is a derivation of x that takes the form $S \Rightarrow^* \alpha A \beta \Rightarrow^* x$ for some $\alpha, \beta \in (\Sigma \cup V)^*$.

A variable that is not useful is *useless*. Clearly, if a variable is either not live or not reachable (see the preceding two exercises), then it is useless. The converse is not true, as the grammar with productions $S \to AB$ and $A \to a$ illustrates. (The variable A is both live and reachable but still useless.)

a. Let G be a CFG. Suppose G_1 is obtained by eliminating all dead variables from G and all productions in which dead variables appear. Suppose G_2 is then obtained from G_1 by eliminating all variables unreachable in G_1, as well as productions in which such variables appear. Show that G_2 contains no useless variables, and $L(G_2) = L(G)$.

Exercise 4.53. (continued)

a. Let G be a CFG. Suppose G_1 is obtained by eliminating all dead variables from G and all productions in which dead variables appear. Suppose G_2 is then obtained from G_1 by eliminating all variables unreachable in G_1, as well as productions in which such variables appear. Show that G_2 contains no useless variables, and $L(G_2) = L(G)$.

b. Give an example to show that if the two steps are done in the opposite order, the resulting grammar may still have useless variables.

Exercise 4.53. (continued)

c. In each case, given the context-free grammar G, find an equivalent CFG with no useless variables.

i. ♣ G has productions

$$S \to ABC \mid BaB \qquad A \to aA \mid BaC \mid aaa$$
$$B \to bBb \mid a \qquad C \to CA \mid AC$$

ii. G has productions

$$S \to AB \mid AC \qquad A \to aAb \mid bAa \mid a \qquad B \to bbA \mid aaB \mid AB$$
$$C \to abCa \mid aDb \qquad D \to bD \mid aC$$

Exercise 4.49. $\clubsuit$

In each case below, find a context-free grammar with no Λ-productions that generates the same language, except possibly for Λ, as the given CFG.

a.

$$S \to AB \mid \Lambda \qquad A \to aASb \mid a \qquad B \to bS$$

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Let G_1 be the context-free grammar with start variabele S and the following productions:

$$S \to Sa \mid bb \mid AB \qquad A \to aAb \mid BBa \qquad B \to SB \mid a \mid \Lambda$$

a. Determine the nullable variabeles in G_1.

b. Give the context-free grammar G_2 resulting from G_1 by eliminating Λ-productions.

c. For each variable X in G_2, give the set of X-derivable variables.

d. Give the context-free grammar G_3 resulting from G_2 by eliminating unit productions.

Exercise 4.50.

In each case, given the context-free grammar G, find a CFG G' in Chomsky normal form that generates the language $L(G) - \{\Lambda\}$.

a. G has productions

$$S \to ABA \qquad A \to aA \mid \Lambda \qquad B \to bB \mid \Lambda$$

b. G has productions

$$S \to aSa \mid bSb \mid \Lambda \qquad A \to aBb \mid bBa \qquad B \to aB \mid bB \mid \Lambda$$

Exercise 4.54.

In each case below, given the context-free grammar G, find a CFG G_1 in Chomsky normal form generating $L(G) - \{\Lambda\}$.

a. ♣ G has productions $S \to SS \mid (S) \mid \Lambda$

b. G has productions $S \to S(S) \mid \Lambda$

c. G has productions

$$S \to AaA \mid CA \mid BaB \qquad A \to aaBa \mid CDA \mid aa \mid DC$$
$$B \to bB \mid bAB \mid bb \mid aS \qquad C \to Ca \mid bC \mid D \qquad D \to bD \mid \Lambda$$

Exercise 4.48. ♣

Show that the nullable variables defined by Definition 4.26 are precisely those variables A for which $A \Rightarrow^* \Lambda$.