From lecture 2:

Theorem

Suppose L is a language over the alphabet Σ. If L is accepted by a finite automaton M, and if n is the number of states of M, then

$\forall$ *for every $x \in L$*
satisfying $|x| \geqslant n$

$\exists$ *there are three strings u, v, and w,*
such that $x = uvw$ and the following three conditions are true:
(1) $|uv| \leqslant n$,
(2) $|v| \geqslant 1$

$\forall$ *and (3) for all $m \geqslant 0$, $uv^m w$ belongs to L*

[M] Thm. 2.29

Example

$L = \{ x \in \{a, b\}^* \mid n_a(x) > n_b(x) \}$ is not accepted by FA

[M] E 2.31

$L \subseteq \{a\}^*$

Example

$L = \{ a^{i^2} \mid i \geqslant 0 \}$ is not accepted by FA

$L = \{\Lambda, a, aaaa, aaaaaaaaa, \ldots\}$

[M] E 2.32

Fun fact

$L^4 = \{a\}^*$

Lagrange's four-square theorem

The length of uv^2w cannot be a square: we will show it is strictly in between two consecutive squares.
$|uv^2w| = |z| + |v| > |z| = n^2$.
$|uv^2w| = |z| + |v| \leqslant n^2 + n < (n+1)^2$.

Let L be the set of legal C programs.
$x = \texttt{main}()\{\{\{\ldots\}\}\}$
[M] E 2.33

Prove the following generalization of the pumping lemma, which can sometimes make it unnecessary to break the proof into cases.

If L can be accepted by an FA,
then there is an integer n
such that for any $x \in L$ with $|x| \geqslant n$
and for any way of writing x as $x_1 x_2 x_3$ with $|x_2| = n$,
there are strings u, v and w such that

a. $x_2 = uvw$
b. $|v| \geqslant 1$
c. For every $m \geqslant 0$, $x_1 u v^m w x_3 \in L$

$L = \{ a^i b^j c^j \mid i \geqslant 1 \text{ and } j \geqslant 0 \} \cup \{ b^j c^k \mid j, k \geqslant 0 \}$

– can be pumped, as in the pumping lemma

– is not accepted by FA

[M] E 2.39

Decision problem: problem for which the answer is 'yes' or 'no':

Given ..., is it true that ...?

Given an undirected graph $G = (V, E)$*,*
does G *contain a Hamiltonian path?*

Given a list of integers $x_1, x_2, \ldots, x_n$*,*
is the list sorted?

decidable $\iff$ $\exists$ algorithm that decides

$M = (Q, \Sigma, \delta, q_0, A)$
membership problem $x \in L(M)$?

Specific to M: Given $x \in \Sigma^*$, is $x \in L(M)$?

Arbitrary M: Given FA M with alphabet Σ, and $x \in \Sigma^*$, is $x \in L(M)$?

Decidable, easy

[M] E 2.34

Theorem

The following two problems are decidable

1. *Given an* FA M, *is* $L(M)$ *nonempty?*
2. *Given an* FA M, *is* $L(M)$ *infinite?*

[M] E 2.34

Lemma

Let M be an FA with n states and let $L = L(M)$.

L is nonempty,
if and only if L contains an element x with $|x| < n$
(at least one such element).

Theorem

The following two problems are decidable

1. *Given an FA M, is $L(M)$ nonempty?*
2. *Given an FA M, is $L(M)$ infinite?*

[M] E 2.34

Lemma

Let M be an FA with n states and let $L = L(M)$.

L is infinite,
if and only if L contains an element x with $|x| \geqslant n$
(at least one such element).

cf. [M] Exercise 2.26

Lemma

Let M be an FA with n states and let $L = L(M)$.

L is infinite,
if and only if L contains an element x with $|x| \geqslant n$
(at least one such element).

Lemma

Let M be an FA with n states and let $L = L(M)$.

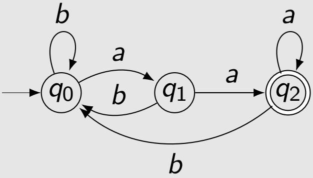
L contains an element x with $|x| \geqslant n$ (at least one such element)
if and only if L contains an element x with $n \leqslant |x| < 2n$
(at least one such element).

- Give 2-state FA for each of the languages over $\{a, b\}$
  - strings with even number of a's
  - strings with at least one b
- Use the product construction to obtain a 4-state FA for the language of strings with even number of a's or at least one b
- Investigate which states can be merged

From lecture 1:

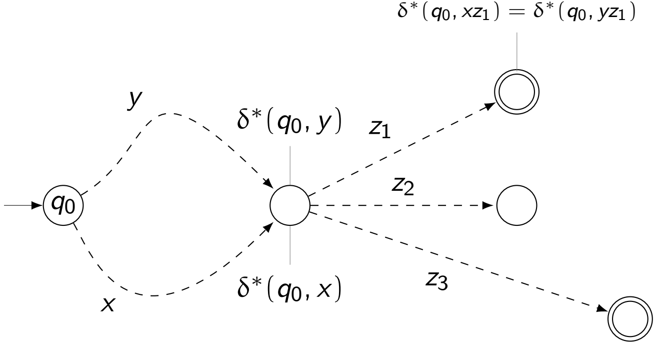
Example

$L_1 = \{\, x \in \{a, b\}^* \mid x \text{ ends with } aa \,\}$



[M] E. 2.1

Same state, same future



Definition

Let L be language over Σ, and let $x, y \in \Sigma^*$.
Then x, y are *distinguishable* wrt L (*L-distinguishable*),
if there exists $z \in \Sigma^*$ with

$$xz \in L \text{ and } yz \notin L \quad \text{or} \quad xz \notin L \text{ and } yz \in L$$

Such z *distinguishes* x and y wrt L.

Equivalent definition:
let $L/x = \{ z \in \Sigma^* \mid xz \in L \}$

x and y are L-*distinguishable* if $L/x \neq L/y$.
Otherwise, they are L-*indistinguishable*.

The strings in a set $S \subseteq \Sigma^*$ are *pairwise L-distinguishable*, if for every pair x, y of distinct strings in S, x and y are L-distinguishable.

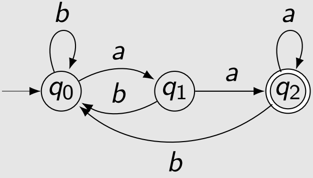
Definition independent of FAs

[M] D 2.20

From lecture 1:

Example

$L_1 = \{ x \in \{a, b\}^* \mid x \text{ ends with } aa \}$



$S = \{\Lambda, a, aa\}$

Example

$L_1 = \{ x \in \{a, b\}^* \mid x \text{ ends with } aa \}$

L/x for $x = \Lambda, a, b, aa \ldots$

Theorem

Suppose $M = (Q, \Sigma, q_0, A, \delta)$ is an FA accepting $L \subseteq \Sigma^*$.

If $x, y \in \Sigma^*$ are L-distinguishable, then $\delta^*(q_0, x) \neq \delta^*(q_0, y)$.

For every $n \geqslant 2$, if there is a set of n pairwise L-distinguishable strings in Σ^*, then Q must contain at least n states.

Hence, indeed: if $\delta^*(q_0, x) = \delta^*(q_0, y)$, then x and y are not L-distinguishable.

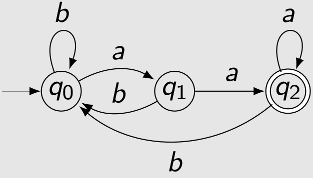
Proof. . .

[M] Thm 2.21

From lecture 1:

Example

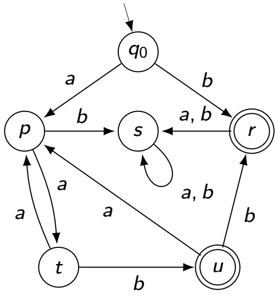
$L_1 = \{\, x \in \{a, b\}^* \mid x \text{ ends with } aa \,\}$



$S = \{\Lambda, a, aa\}$

$L = \{aa, aab\}^*\{b\}$
[M] E 2.22

$L = \{aa, aab\}^*\{b\}$



[M] E 2.22