From lecture 8:

Definition

A derivation in a context-free grammar is a *leftmost* derivation, if at each step, a production is applied to the leftmost variable-occurrence in the current string.
A *rightmost* derivation is defined similarly.

[M] D 4.16

derivation step $\alpha = \alpha_1 A \alpha_2 \Rightarrow_G \alpha_1 \gamma \alpha_2 = \beta$ for $A \to \gamma \in P$

The derivation step is *leftmost* iff $\alpha_1 \in \Sigma^*$

We write $\alpha \overset{\ell}{\Rightarrow} \beta$

From lecture 8:

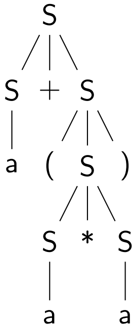
$S \to a \mid S + S \mid S * S \mid (S) \qquad \Sigma = \{a, +, *, (,)\}$

$S \Rightarrow S + \underline{S} \Rightarrow S + (\underline{S}) \Rightarrow S + (\underline{S} * S) \Rightarrow$
$\underline{S} + (a * S) \Rightarrow a + (a * \underline{S}) \Rightarrow a + (a * a)$

Derivation tree...

[M] E 4.2, Fig 4.15

Expressions

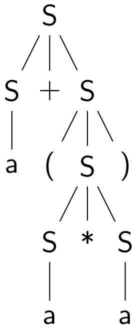


[M] E 4.2, Fig 4.15

$S \to a \mid S + S \mid S * S \mid (S) \quad \Sigma = \{a, +, *, (,)\}$

$S \Rightarrow S + \underline{S} \Rightarrow S + (\underline{S}) \Rightarrow S + (\underline{S} * S) \Rightarrow \underline{S} + (a * S) \Rightarrow a + (a * \underline{S}) \Rightarrow a + (a * a)$

Leftmost derivation...



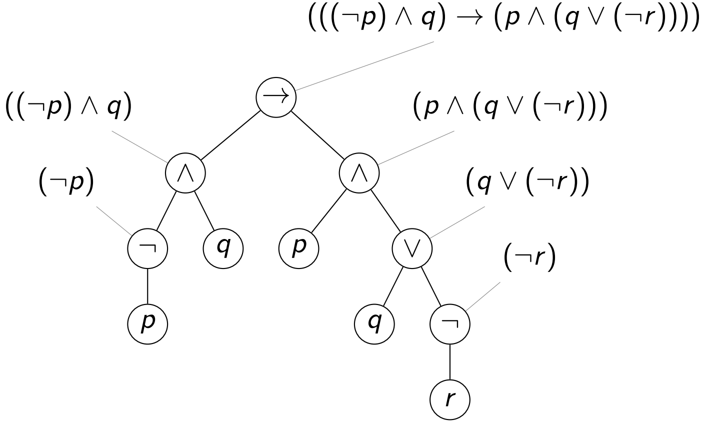
$S \to a \mid S + S \mid S * S \mid (S) \quad \Sigma = \{a, +, *, (,)\}$

$S \Rightarrow S + \underline{S} \Rightarrow S + (\underline{S}) \Rightarrow S + (\underline{S} * S) \Rightarrow \underline{S} + (a * S) \Rightarrow a + (a * \underline{S}) \Rightarrow a + (a * a)$

Leftmost derivation:
$S \overset{\ell}{\Rightarrow} \underline{S} + S \overset{\ell}{\Rightarrow} a + S \overset{\ell}{\Rightarrow} a + (\underline{S}) \overset{\ell}{\Rightarrow} a + (\underline{S} * S) \overset{\ell}{\Rightarrow} a + (a * \underline{S}) \overset{\ell}{\Rightarrow} a + (a * a)$

One tree $\leftrightarrow$ multiple derivations
Tree $\to$ derivation $\to$ (same) tree
[M] E 4.2, Fig 4.15

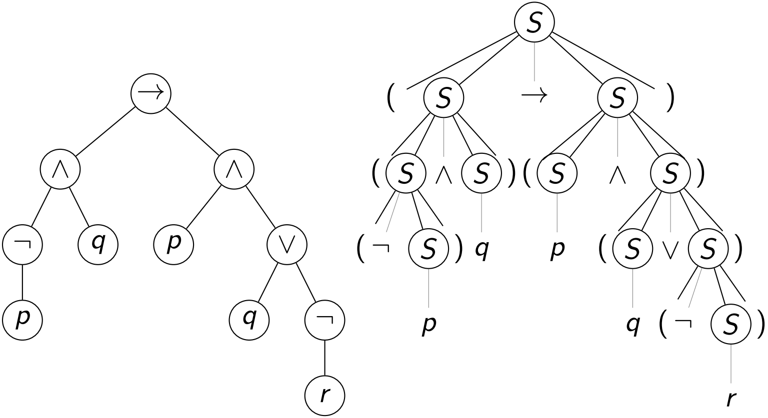
$\psi ::= p \mid (\neg\psi) \mid (\psi \wedge \psi) \mid (\psi \vee \psi) \mid (\psi \rightarrow \psi)$



[H&R] Fig 1.3

$S ::= p \mid q \mid r \mid (\neg S) \mid (S \wedge S) \mid (S \vee S) \mid (S \rightarrow S)$

parse tree vs. derivation tree[1]

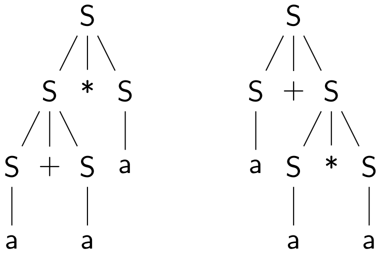


[1] with all brackets explicit

Definition

A context-free grammar G is *ambiguous*, if for at least one $x \in L(G)$, x has more than one derivation tree.

Otherwise: *unambiguous* [M] D 4.18



$$\Sigma = \{a, +, *, (,)\}$$
$$S \to a \mid S + S \mid S * S \mid (S)$$

$a + a * a$

leftmost derivation $\longleftrightarrow$ derivation tree

Theorem

If G is a context-free grammar, then for every $x \in L(G)$, these three statements are equivalent:

1. *x has more than one derivation tree*
2. *x has more than one leftmost derivation*
3. *x has more than one rightmost derivation*

Proof. . .

[M] Thm 4.17

leftmost derivation $\longleftrightarrow$ derivation tree

Theorem

If G is a context-free grammar, then for every $x \in L(G)$, these three statements are equivalent:

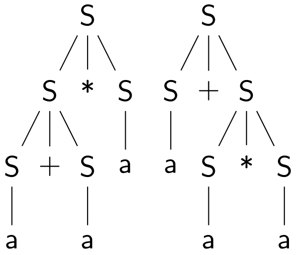
1. *x has more than one derivation tree*
2. *x has more than one leftmost derivation*
3. *x has more than one rightmost derivation*

[M] Thm 4.17

Definition

A context-free grammar G is *ambiguous*, if for at least one $x \in L(G)$, x has more than one derivation tree (or, equivalently, more than one leftmost derivation).

Otherwise: *unambiguous* [M] D 4.18



$\Sigma = \{a, +, *, (,)\}$

$S \to a \mid S + S \mid S * S \mid (S)$

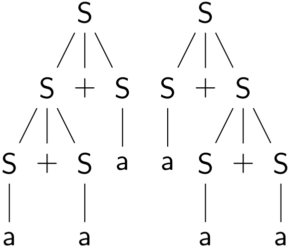
$a + a * a$

$S \overset{\ell}{\Rightarrow} \underline{S} * S \overset{\ell}{\Rightarrow} S + S * S \overset{\ell}{\Rightarrow} a + S * S \overset{\ell}{\Rightarrow} a + a * S \overset{\ell}{\Rightarrow} a + a * a$

$S \overset{\ell}{\Rightarrow} \underline{S}{+}S \overset{\ell}{\Rightarrow} a{+}S \overset{\ell}{\Rightarrow} a{+}S{*}S \overset{\ell}{\Rightarrow} a{+}a{*}S \overset{\ell}{\Rightarrow} a + a * a$

leftmost derivation $\longleftrightarrow$ derivation tree

Ambiguity (2)



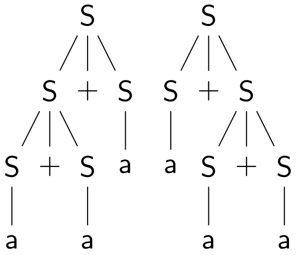
$\Sigma = \{a, +, *, (,)\}$
$S \to a \mid S + S \mid S * S \mid (S)$

$a + a + a$

Leftmost for 1:
$S \overset{\ell}{\Rightarrow} \underline{S} + S \overset{\ell}{\Rightarrow} S + S + S \overset{\ell}{\Rightarrow} a + S + S \overset{\ell}{\Rightarrow}$
$a + a + S \overset{\ell}{\Rightarrow} a + a + a$

Derivation for 2:
$S \Rightarrow S + \underline{S} \Rightarrow S + S + S \Rightarrow a + S + S \Rightarrow$
$a + a + S \Rightarrow a + a + a$

Ambiguity (2)



$\Sigma = \{a, +, *, (,)\}$
$S \to a \mid S + S \mid S * S \mid (S)$

$a + a + a$

Leftmost for 1:
$S \overset{\ell}{\Rightarrow} \underline{S} + S \overset{\ell}{\Rightarrow} S + S + S \overset{\ell}{\Rightarrow} a + S + S \overset{\ell}{\Rightarrow}$
$a + a + S \overset{\ell}{\Rightarrow} a + a + a$

Derivation for 2:
$S \Rightarrow S + \underline{S} \Rightarrow S + S + S \Rightarrow a + S + S \Rightarrow$
$a + a + S \Rightarrow a + a + a$

Leftmost for 2:
$S \overset{\ell}{\Rightarrow} \underline{S} + S \overset{\ell}{\Rightarrow} a + S \overset{\ell}{\Rightarrow} a + S + S \overset{\ell}{\Rightarrow}$
$a + a + S \overset{\ell}{\Rightarrow} a + a + a$

leftmost derivation $\longleftrightarrow$ derivation tree

This example is a little weird. In the derivation step $\mathsf{S} + \mathsf{S} \Rightarrow \mathsf{S} + \mathsf{S} + \mathsf{S}$ we cannot really see which S has been rewritten.

Expr

ambiguous:

$S \rightarrow a \mid S + S \mid S * S \mid (S)$

[M] E 4.20

$a + a * a$

unambiguous:

...

Expr

ambiguous:

$S \to a \mid S + S \mid S * S \mid (S)$

[M] E 4.20

$a + a * a$

unambiguous:

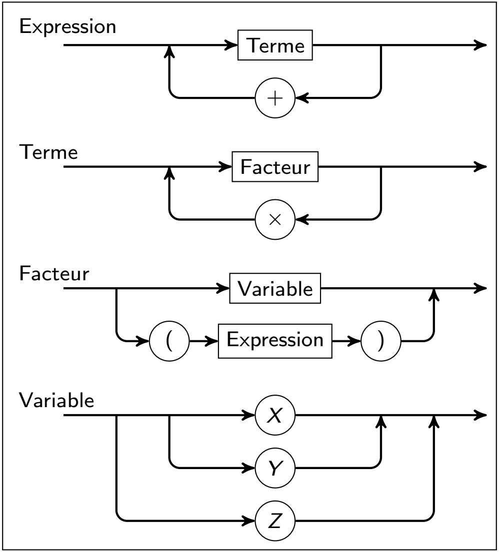
$S \to S + T \mid T$

$T \to T * F \mid F$

$F \to a \mid (S)$

[M] Thm 4.25

The proof of the unambiguity of this grammar does not have to be known for the exam



right associative

http://math.et.info.free.fr/TikZ/index.html

Chapitre 7

$AeqB = \{ x \in \{a, b\}^* \mid n_a(x) = n_b(x) \}$

$aaabbb, ababab, aababb, \ldots$

$$\begin{aligned} S &\to \Lambda \mid aB \mid bA \\ A &\to aS \mid bAA \\ B &\to bS \mid aBB \end{aligned}$$

A generates $n_a(x) = n_b(x) + 1$

B generates $n_a(x) + 1 = n_b(x)$

Derivation for $aababb$:

$S \Rightarrow aB \Rightarrow aaBB \Rightarrow aabSB \Rightarrow \ldots$ (different options)

(1) $aabB \Rightarrow aabaBB \Rightarrow aababSB \Rightarrow aababB \Rightarrow aababbS \Rightarrow aababb$

(2) $aaba\underline{B}B \Rightarrow aabab\underline{S}B \Rightarrow aabab\underline{B} \Rightarrow aababb\underline{S} \Rightarrow aababb$

(2') $aabaB\underline{B} \Rightarrow aaba\underline{B}bS \Rightarrow aababSb\underline{S} \Rightarrow aabab\underline{S}b \Rightarrow aababb$

[M] E 4.8

ABOVE

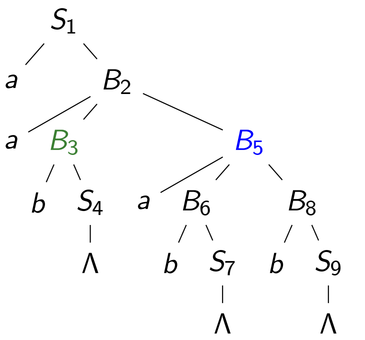
When a string has multiple variables, like $aabSB$ in the above example, then we are not forced to rewrite the first variable, we can as well rewrite another one.

Thus we can do $aab\underline{S}B \Rightarrow aabB$, but also $aabS\underline{B} \Rightarrow aabSaBB$, for instance.

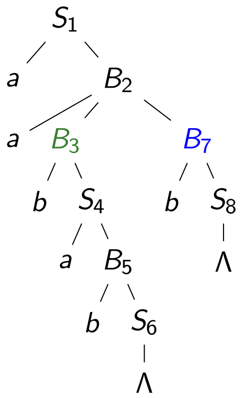
BELOW

In detail, two different derivation trees for the same string, corresponding to derivations (1) and (2,2') respectively, together with two associated leftmost derivations.

Given these two trees we conclude the grammar is ambiguous.



$S \Rightarrow aB \Rightarrow aaBB \Rightarrow aabSB \Rightarrow aabB \Rightarrow aabaBB \Rightarrow aababSB \Rightarrow aababB \Rightarrow aababbS \Rightarrow aababb$



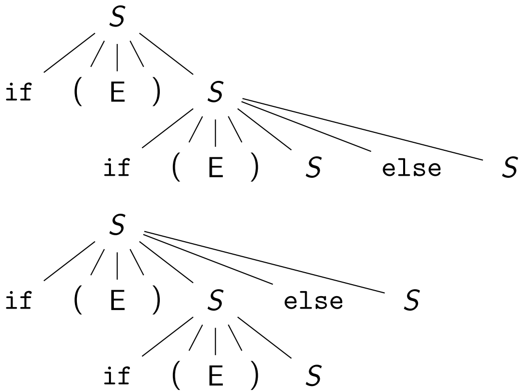
$S \Rightarrow aB \Rightarrow aaBB \Rightarrow aabSB \Rightarrow aabaBB \Rightarrow aababSB \Rightarrow aababB \Rightarrow aababbS \Rightarrow aababb$

$S \rightarrow \texttt{if} \; (\; E \;) \; S \; \mid \; \texttt{if} \; (\; E \;) \; S \; \texttt{else} \; S \; \mid \; \ldots$

$\texttt{if} \; (\; E \;) \; \texttt{if} \; (\; E \;) \; S \; \texttt{else} \; S$

[M] E 4.19

$S \to$ if (E) S | if (E) S else S | ...



[M] E 4.19

ambiguous:

$S \to \texttt{if}\ (\ E\ )\ S \mid \texttt{if}\ (\ E\ )\ S\ \texttt{else}\ S \mid A \mid \ \ldots$

unambiguous...

[M] E 4.19

ambiguous:

$S \to \texttt{if}\ (\ E\ )\ S \mid \texttt{if}\ (\ E\ )\ S\ \texttt{else}\ S \mid A \mid \ \dots$

unambiguous:

$S \to S_1 \mid S_2$

$S_1 \to \texttt{if}\ (\ E\ )\ S_1\ \texttt{else}\ S_1 \mid A \mid \ \dots$ (matched)

$S_2 \to \texttt{if}\ (\ E\ )\ S \mid \texttt{if}\ (\ E\ )\ S_1\ \texttt{else}\ S_2$ (open)

[M] E 4.19

Balanced

ambiguous:

$S \to SS \mid (S) \mid \Lambda$ (more or less the definition of balanced)

unambiguous:

$S \to (S)S \mid \Lambda$

[M] Exercise 4.45

Ambiguous

Some cf languages are inherently ambiguous

Ambiguity is *undecidable*

[M] Theorem 9.20