Results homework 1...

reg. languages	FA	reg. grammar	reg. expression
determ. cf. languages	DPDA		
cf. languages	PDA	cf. grammar	
cs. languages	LBA	cs. grammar	
re. languages	TM	unrestr. grammar	

From lecture 7:

$AnBn = \{ a^n b^n \mid n \geq 0 \} \subseteq \{a, b\}^*$

Example

– $\Lambda \in AnBn$ (basis)

– for every $x \in AnBn$, also $axb \in AnBn$ (induction)

$S \to \Lambda$

$S \to aSb$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aa\,bb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaa\,bbb$$

if $S \Rightarrow^* x$ then also $S \Rightarrow^* axb$

From lecture 7:

Example

– $\Lambda, a, b \in Pal$
– for every $x \in Pal$, also $axa, bxb \in Pal$

$S \to \Lambda \mid a \mid b$
$S \to aSa$
$S \to bSb$

$S \Rightarrow aSa \Rightarrow aaSaa \Rightarrow aabSbaa \Rightarrow aababaa$

$NonPal \subseteq \{a, b\}^*$
$x = abbbaaba \in NonPal$
[M] E 4.3

Non palindromes

$NonPal \subseteq \{a, b\}^*$
$x = abbbaaba \in NonPal$

Example

– for every S in $NonPal$, aSa and bSb are in $NonPal$
– for every $A \in \{a, b\}^*$, aAb and bAa are elements of $NonPal$

$S \to aAb \mid bAa \mid aSa \mid bSb$
$A \to \Lambda \mid aA \mid bA$
[M] E 4.3

$NonPal \subseteq \{a, b\}^*$
$x = abbbaaba \in NonPal$

Example

– for every S in $NonPal$, aSa, bSb, aSb and bSa are in $NonPal$
– for every $A \in \{a, b\}^*$, aAb and bAa are elements of $NonPal$

$S \to aAb \mid bAa \mid aSa \mid bSb \mid aSb \mid bSa$
$A \to \Lambda \mid aA \mid bA$
[M] E 4.3

From lecture 7:

$L_1 = \{ a^i b^j c^k \mid i = j + k \}$ $\quad aaa\,b\,cc$

generate as $\quad a^{k+j}\ b^j\ c^k = \underbrace{a^k\ \underbrace{a^j\ b^j}\ c^k}$

$$S \to aSc \mid T$$
$$T \to aTb \mid \Lambda$$

$$S \Rightarrow aSc \Rightarrow aaScc \Rightarrow aaTcc \Rightarrow aaaTbcc \Rightarrow aaabcc$$

$L_2 = \{ a^i b^j c^k \mid j = i + k \}$ $\quad a\,bbb\,cc$

generate as $\quad a^i\ b^{i+k}\ c^k = \underbrace{a^i\ b^i}\ \underbrace{b^k\ c^k}$

$$S \to XY \qquad \text{(concatenate)}$$
$$X \to aXb \mid \Lambda$$
$$Y \to bYc \mid \Lambda$$

$$S \Rightarrow \underline{X}\ Y \Rightarrow a\underline{X}b\ Y \Rightarrow ab\ \underline{Y} \Rightarrow ab\ b\underline{Y}c \Rightarrow ab\ bb\underline{Y}cc \Rightarrow abbbcc$$
$$S \Rightarrow X\ \underline{Y} \Rightarrow \underline{X}\ bYc \Rightarrow aXb\ b\underline{Y}c \Rightarrow a\underline{X}b\ bbYcc \Rightarrow ab\ bb\underline{Y}cc \Rightarrow abbbcc$$

(a priori there is no prescribed order rewriting X or Y)

$$L_0 = \{ a^i b^j c^k \mid j = i + k \} = \{ a^i b^{i+k} c^k \mid i, k \geq 0 \}$$
$$= \{ \underbrace{a^i b^i}\underbrace{b^k c^k} \mid i, k \geq 0 \}$$

$$S_0 \to XY \quad X \to aXb \mid \Lambda \quad Y \to bYc \mid \Lambda$$

$$L = \{ a^i b^j c^k \mid j \neq i + k \}$$

[M] E 4.10

$$L_0 = \{\ a^i b^j c^k \mid j = i + k\ \} = \{\ a^i b^{i+k} c^k \mid i, k \geq 0\ \}$$
$$= \{\ \underbrace{a^i b^i}\underbrace{b^k c^k} \mid i, k \geq 0\ \}$$

$$S_0 \to XY \qquad X \to aXb \mid \Lambda \qquad Y \to bYc \mid \Lambda$$

$L = \{\ a^i b^j c^k \mid j \neq i + k\ \} = L_1 \cup L_2$
$S \to S_1 \mid S_2$

$L_1 = \{\ a^i b^j c^k \mid j > i + k\ \}$
$L_2 = \{\ a^i b^j c^k \mid j < i + k\ \}$

[M] E 4.10

$$L_0 = \{ a^i b^j c^k \mid j = i+k \} = \{ a^i b^{i+k} c^k \mid i,k \geq 0 \}$$
$$= \{ \underbrace{a^i b^i}\underbrace{b^k c^k} \mid i,k \geq 0 \}$$

$$S_0 \to XY \quad X \to aXb \mid \Lambda \quad Y \to bYc \mid \Lambda$$

$L = \{ a^i b^j c^k \mid j \neq i+k \} = L_1 \cup L_2$
$S \to S_1 \mid S_2$

$L_1 = \{ a^i b^j c^k \mid j > i+k \}$

$S_1 \to X_1 b Y_1$
$X_1 \to a X_1 b \mid X_1 b \mid \Lambda$
$Y_1 \to b Y_1 c \mid b Y_1 \mid \Lambda$

$L_2 = \{ a^i b^j c^k \mid j < i+k \}$

[M] E 4.10

$$L_0 = \{ a^i b^j c^k \mid j = i + k \} = \{ a^i b^{i+k} c^k \mid i, k \geq 0 \}$$
$$= \{ \underbrace{a^i b^i}\underbrace{b^k c^k} \mid i, k \geq 0 \}$$

$$S_0 \to XY \quad X \to aXb \mid \Lambda \quad Y \to bYc \mid \Lambda$$

$L = \{ a^i b^j c^k \mid j \neq i + k \} = L_1 \cup L_2$

$S \to S_1 \mid S_2$

$L_1 = \{ a^i b^j c^k \mid j > i + k \}$

$S_1 \to X_1 b Y_1$

$X_1 \to aX_1 b \mid X_1 b \mid \Lambda$

$Y_1 \to bY_1 c \mid bY_1 \mid \Lambda$

$L_2 = \{ a^i b^j c^k \mid j < i + k \}$

$S_2 \to aX_2 Y_2 \mid X_2 Y_2 c$

$X_2 \to aX_2 b \mid aX_2 \mid \Lambda$

$Y_2 \to bY_2 c \mid Y_2 c \mid \Lambda$

[M] E 4.10

ABOVE

The solution in the book is a bit complex. We have made it shorter here.

From lecture 7:

Definition

context-free grammar (CFG) 4-tuple $G = (V, \Sigma, S, P)$
- V alphabet *variables* / *nonterminals*
- Σ alphabet *terminals* disjoint $V \cap \Sigma = \varnothing$
- $S \in V$ *axiom*, *start symbol*
- P finite set rules, *productions*
  of the form $A \to \alpha$, $A \in V$, $\alpha \in (V \cup \Sigma)^*$

derivation step $\alpha = \alpha_1 A \alpha_2 \Rightarrow_G \alpha_1 \gamma \alpha_2 = \beta$ for $A \to \gamma \in P$

Definition

language generated by G
$L(G) = \{ x \in \Sigma^* \mid S \Rightarrow_G^* x \}$

[M] Def 4.6 & 4.7

From lecture 7:

Using building blocks

Theorem

If L_1, L_2 are CFL, then so are $L_1 \cup L_2$, $L_1 L_2$ and L_1^*.

$G_i = (V_i, \Sigma, S_i, P_i)$, having no variables in common.

Construction

$G = (V_1 \cup V_2 \cup \{S\}, \Sigma, S, P)$, new axiom S

– $P = P_1 \cup P_2 \cup \{S \to S_1, S \to S_2\}$ $L(G) = L(G_1) \cup L(G_2)$

– $P = P_1 \cup P_2 \cup \{S \to S_1 S_2\}$ $L(G) = L(G_1)L(G_2)$

$G = (V_1 \cup \{S\}, \Sigma, S, P)$, new axiom S

– $P = P_1 \cup \{S \to SS_1, S \to \Lambda\}$ $L(G) = L(G_1)^*$

Proof. . .

[M] Thm 4.9

Regular languages are closed under

– Boolean operations (complement, union, intersection, minus)
– Regular operations (union, concatenation, star)
– Reverse (mirror)
– [inverse] Homomorphism

Fact, proof follows ↪later

Theorem

the languages
– $AnBnCn = \{ a^n b^n c^n \mid n \geq 0 \}$ *and*
– $XX = \{xx \mid x \in \{a, b\}^* \}$
are not context-free

[M] E 6.3, E 6.4

AnBnCn is the intersection of two context-free languages
[M] E 6.10

The complement of both *AnBnCn* and *XX* is context-free.
[M] E 6.11
Hence, CFL is not closed under intersection, complement

$S \to S_1 \mid S_2$ union
$S \to S_1 S_2$ concatenation
$S \to SS_1 \mid \Lambda$ star

CFG for $\varnothing$...
CFG for $\{\sigma\}$...

Example

$L = bba(ab)^* + (ab + ba^*b)^*ba$

[M] E 4.11

$S \to S_1 \mid S_2$	union
$S \to S_1 S_2$	concatenation
$S \to S S_1 \mid \Lambda$	star

Example

$L = bba(ab)^* + (ab + ba^*b)^*ba$

$S \to S_1 \mid S_2$

$S_1 \to S_1ab \mid bba$

$S_2 \to TS_2 \mid ba \quad T \to ab \mid bUb \quad U \to aU \mid \Lambda$

[M] E 4.11

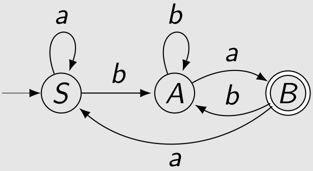
ABOVE

We have seen constructions to apply the regular operations (union, concatenation and star) to context-free grammars. These we can now use to build CFG for regular expressions.

There is a better way to build CFG for regular languages. Use finite automata, and simulate these using a very simple type of context-free grammar. These simple grammars are called regular.

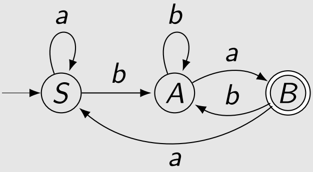
systematic approach

Example



systematic approach

Example



axiom S — initial state

$S \to bA \mid aS$ — transitions

$A \to bA \mid aB$

$B \to bA \mid aS$

$B \to \Lambda$ — accepting state

path / derivation for *bbaaba*...

Definition

regular grammar (or *right-linear grammar*)
productions are of the form
– $A \to \sigma B$ variables A, B, terminal σ
– $A \to \Lambda$ variable A

Special type of context-free grammar

Theorem

A language L is regular,
if and only if there is a regular grammar generating L.

Proof. . .

[M] Def 4.13, Thm 4.14

4.4 Derivation trees and ambiguity

A derivation...

$S \to a \mid S + S \mid S * S \mid (S) \quad \Sigma = \{a, +, *, (,)\}$

$S \Rightarrow S + \underline{S} \Rightarrow S + (\underline{S}) \Rightarrow S + (\underline{S} * S) \Rightarrow$
$\underline{S} + (a * S) \Rightarrow a + (a * \underline{S}) \Rightarrow a + (a * a)$

[M] E 4.2, Fig 4.15

Definition

A derivation in a context-free grammar is a *leftmost* derivation, if at each step, a production is applied to the leftmost variable-occurrence in the current string.
A *rightmost* derivation is defined similarly.

[M] D 4.16

derivation step $\alpha = \alpha_1 A \alpha_2 \Rightarrow_G \alpha_1 \gamma \alpha_2 = \beta$ for $A \to \gamma \in P$

The derivation step is *leftmost* iff $\alpha_1 \in \Sigma^*$

We write $\alpha \overset{\ell}{\Rightarrow} \beta$

$S \to a \mid S + S \mid S * S \mid (S) \quad \Sigma = \{a, +, *, (,)\}$

$S \Rightarrow S + \underline{S} \Rightarrow S + (\underline{S}) \Rightarrow S + (\underline{S} * S) \Rightarrow$
$\underline{S} + (a * S) \Rightarrow a + (a * \underline{S}) \Rightarrow a + (a * a)$

Derivation tree...

[M] E 4.2, Fig 4.15