Automata Theory

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⟨assignment⟩ ::= ⟨variable⟩ = ⟨expression⟩

⟨statement⟩ ::= ⟨assignment⟩ |
    ⟨compound-statement⟩ |
    ⟨if-statement⟩ |
    ⟨while-statement⟩ | . . .

⟨if-statement⟩ ::=
    if ⟨test⟩ then ⟨statement⟩ |
    if ⟨test⟩ then ⟨statement⟩ else ⟨statement⟩

⟨while-statement⟩ ::=
    while ⟨test⟩ do ⟨statement⟩

Definition (well-formed formulas)

...by using the construction rules below, and only those, finitely many times:

– every propositional atom $p, q, r, \ldots$ is a wff

– if ϕ is a wff, then so is $(\neg\phi)$

– if ϕ and ψ are wff, then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$,

BNF   Backus Naur form

$\psi ::= p \mid (\neg\psi) \mid (\psi \wedge \psi) \mid (\psi \vee \psi) \mid (\psi \to \psi)$

M.Huet & M.Ryan, Logic in Computer Science

Example

– $\Lambda \in AnBn$ (basis)
– for every $x \in AnBn$, also $axb \in AnBn$ (induction)

$S \to \Lambda$
$S \to aSb$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aa\,bb$$
$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaa\,bbb$$

if $S \Rightarrow^* x$ then also $S \Rightarrow^* axb$

Example

– $\Lambda, a, b \in Pal$
– for every $x \in Pal$, also $axa, bxb \in Pal$

$S \to \Lambda \mid a \mid b$
$S \to aSa$
$S \to bSb$

$S \Rightarrow aSa \Rightarrow aaSaa \Rightarrow aabSbaa \Rightarrow aababaa$

$AnBn = \{ a^n b^n \mid n \geqslant 0 \}$

variants

$\{ a^n b^{n+1} \mid n \geqslant 0 \}$

$S \to b$ (end with extra b)

$S \to aSb$

$\{ a^i b^j \mid i \leqslant j \}$

$S \to \Lambda$

$S \to aSb \mid Sb$ (free b's)

$\{ a^i b^j \mid i \neq j \}$

$S \to A \mid B$ (choice!)

$A \to aAb \mid aA \mid a$ $(i > j)$

$B \to aBb \mid Bb \mid b$ $(i < j)$

$Balanced \subseteq \{(,)\}^*$: $\Lambda, (), (()), ()(), ((())), (())(), \ldots$

Example

- $\Lambda \in Balanced$
- for every $x, y \in Balanced$, also $xy \in Balanced$
- for every $x \in Balanced$, also $(x) \in Balanced$

$S \to \Lambda \mid SS \mid (S)$

$S \Rightarrow SS \Rightarrow (S)S \Rightarrow ()S \Rightarrow ()(S) \Rightarrow ()((S)) \Rightarrow ()(())$

Example

– $a \in \mathit{Expr}$
– for every $x, y \in \mathit{Expr}$, also $x + y \in \mathit{Expr}$ and $x * y \in \mathit{Expr}$
– for every $x \in \mathit{Expr}$, also $(x) \in \mathit{Expr}$

$$S \to a \mid S + S \mid S * S \mid (S)$$

$$S \Rightarrow S + S \to S + S * S$$
$$S \Rightarrow S * S \to S + S * S$$

ambiguity. . .

[M] E 4.2

Non palindromes

$NonPal \subseteq \{a, b\}^*$
$x = abbbaaba \in NonPal$

Example

– for every $A \in \{a, b\}^*$, aAb and bAa are elements of $NonPal$
– for every S in $NonPal$, aSa, bSb are in $NonPal$

$A \to \Lambda \mid aA \mid bA$
$S \to aAb \mid bAa \mid aSa \mid bSb$
[M] E 4.3

alphabet $\{ 1, 2, 5, = \}$

$\{ x{=}y \mid x \in \{1, 2\}^*, y \in \{5\}^*, n_1(x) + 2n_2(x) = 5n_5(y) \}$

$n_\sigma(x)$ number of σ occurrences in x

212=5   22222=55   121221221222=5555

variables S_i, $0 \leqslant i \leqslant 4$

axiom S_0

productions

$S_0 \to 1S_1 \mid 2S_2$
$S_1 \to 1S_2 \mid 2S_3$
$S_2 \to 1S_3 \mid 2S_4$
$S_3 \to 1S_4 \mid 2S_05$
$S_4 \to 1S_05 \mid 2S_15$
$S_0 \to =$

Definition

context-free grammar (CFG) 4-tuple $G = (V, \Sigma, S, P)$

- V alphabet *variables* / *nonterminals*
- Σ alphabet *terminals* disjoint $V \cap \Sigma = \varnothing$
- $S \in V$ *axiom*, *start symbol*
- P finite set rules, *productions*
  of the form $A \to \alpha$, $A \in V$, $\alpha \in (V \cup \Sigma)^*$

derivation step $\alpha = \alpha_1 A \alpha_2 \Rightarrow_G \alpha_1 \gamma \alpha_2 = \beta$ for $A \to \gamma \in P$

Definition

language generated by G

$L(G) = \{ x \in \Sigma^* \mid S \Rightarrow_G^* x \}$

[M] Def 4.6 & 4.7

NonPal, its grammar components

$$A \to \Lambda \mid aA \mid bA$$
$$S \to aAb \mid bAa \mid aSa \mid bSb$$

variables $V = \{\, S, A \,\}$
terminals $\Sigma = \{\, a, b \,\}$
axiom S
productions
$P = \{A \to \Lambda, A \to aA, A \to bA, S \to aAb, S \to bAa, S \to aSa, S \to bSb\}$

$\Rightarrow_G^*$ is the *transitive and reflexive closure* of $\Rightarrow_G$

zero, one or more steps

general case $\quad \alpha = \alpha_0 \Rightarrow \alpha_1 \Rightarrow \ldots \Rightarrow \alpha_n = \beta$

$\alpha \Rightarrow_G^* \beta$ iff there are strings $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that
- $\alpha_0 = \alpha$
- $\alpha_n = \beta$
- $\alpha_i \Rightarrow \alpha_{i+1} \quad$ for $0 \leqslant i < n$.

special case $\quad n = 0 \quad \alpha = \alpha_0 = \beta$

Variables can be rewritten regardless of context

Lemma

If $u_1 \Rightarrow^* v_1$ *and* $u_2 \Rightarrow^* v_2$*, then* $u_1 u_2 \Rightarrow^* v_1 v_2$*.*

Lemma

If $u \Rightarrow^* v_1 v v_2$ *and* $v \Rightarrow^* w$*, then* $u \Rightarrow^* v_1 w v_2$*.*

Lemma

If $u \Rightarrow^* v$ *and* $u = u_1 u_2$*,*
then $v = v_1 v_2$ *such that* $u_1 \Rightarrow^* v_1$ *and* $u_2 \Rightarrow^* v_2$*.*

$AeqB = \{\ x \in \{a, b\}^* \mid n_a(x) = n_b(x)\ \}$

$aaabbb, ababab, aababb, \ldots$

$$\begin{aligned} S &\to \Lambda \mid aB \mid bA \\ A &\to aS \mid bAA \\ B &\to bS \mid aBB \end{aligned}$$

A generates $n_a(x) = n_b(x) + 1$
B generates $n_a(x) + 1 = n_b(x)$

$S \Rightarrow aB \Rightarrow aaBB \Rightarrow aabSB \Rightarrow \ldots$ (different options)

$\ldots aabB \Rightarrow aabaBB \Rightarrow aababSB \Rightarrow aababB \Rightarrow aababbS \Rightarrow aababb$
$\ldots aabSaBB \Rightarrow aabSabSB \Rightarrow aabSabB \Rightarrow aabSabbS \Rightarrow aabSabb \Rightarrow aababb$

[M] E 4.8

$L_1 = \{ a^i b^j c^k \mid i = j + k \}$ $\quad aaa\,b\,cc$

generate as $\quad a^{k+j}\ b^j\ c^k = \underbrace{a^k\ \underbrace{a^j\ b^j}\ c^k}$

$S \to aSc \mid T$
$T \to aTb \mid \Lambda$

$S \Rightarrow aSc \Rightarrow aaScc \Rightarrow aaTcc \Rightarrow aaaTbcc \Rightarrow aaabcc$

$L_2 = \{ a^i b^j c^k \mid j = i + k \}$ $\quad a\,bbb\,cc$

generate as $\quad a^i\ b^{i+k}\ c^k = \underbrace{a^i\ b^i}\ \underbrace{b^k\ c^k}$

$S \to XY$ (concatenate)
$X \to aXb \mid \Lambda$
$Y \to bYc \mid \Lambda$

$S \Rightarrow \underline{X}\ Y \Rightarrow a\underline{X}b\ Y \Rightarrow ab\ \underline{Y} \Rightarrow ab\ b\underline{Y}c \Rightarrow ab\ bb\underline{Y}cc \Rightarrow abbbcc$
$S \Rightarrow X\ \underline{Y} \Rightarrow \underline{X}\ bYc \Rightarrow aXb\ b\underline{Y}c \Rightarrow a\underline{X}b\ bbYcc \Rightarrow ab\ bb\underline{Y}cc \Rightarrow$
$abbbcc$

Using building blocks

Theorem

If L_1, L_2 are CFL, then so are $L_1 \cup L_2$, $L_1 L_2$ and L_1^*.

$G_i = (V_i, \Sigma, S_i, P_i)$, having no variables in common.

Construction

$G = (V_1 \cup V_2 \cup \{S\}, \Sigma, S, P)$, new axiom S
– $P = P_1 \cup P_2 \cup \{S \to S_1, S \to S_2\}$ $L(G) = L(G_1) \cup L(G_2)$
– $P = P_1 \cup P_2 \cup \{S \to S_1 S_2\}$ $L(G) = L(G_1) L(G_2)$

$G = (V_1 \cup \{S\}, \Sigma, S, P)$, new axiom S
– $P = P_1 \cup \{S \to SS_1, S \to \Lambda\}$ $L(G) = L(G_1)^*$

[M] Thm 4.9