Automata Theory

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Bachelor Informatica
Data Science and Artificial Intelligence
Universiteit Leiden

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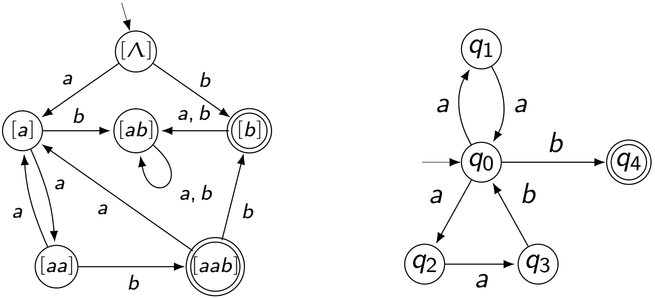
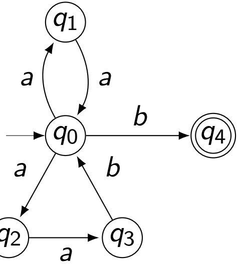
Universiteit Leiden
Leiden Institute of
Advanced Computer Science

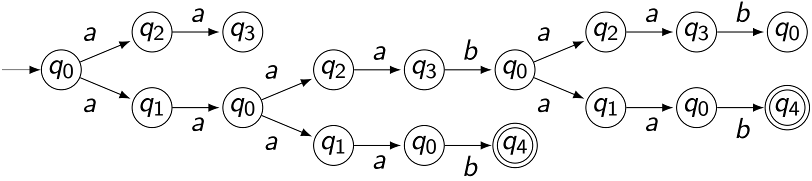
$L = \{aa, aab\}^*\{b\}$



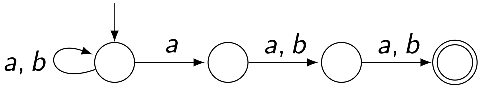
[M] E 3.6



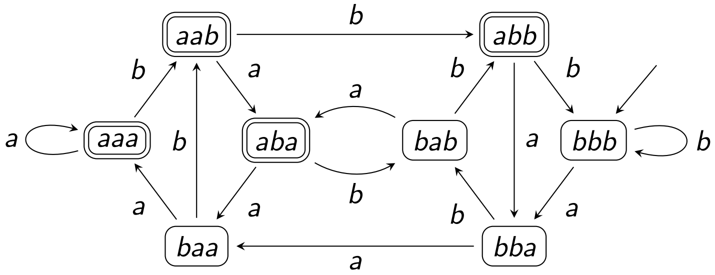
$x = aaaabaab$



[M] E 3.6. also ↪E 2.22



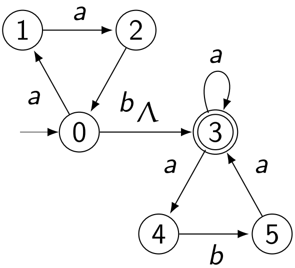
Also ↪deterministic



$n + 1$ versus 2^n states.

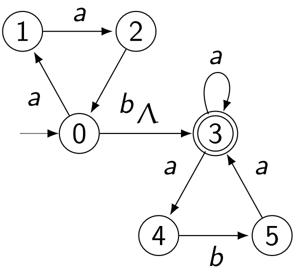
[M] E. 2.24

$L = \{aab\}^*\{a, aba\}^*$

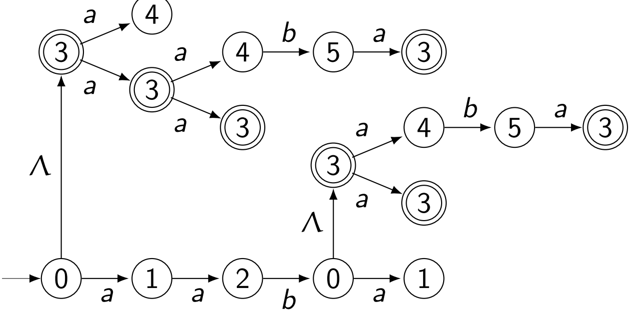


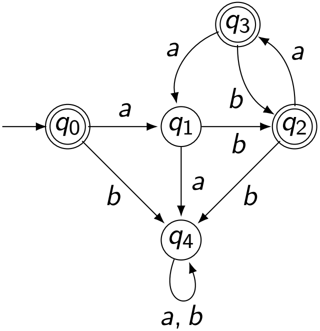
NFA

[M] E 3.9

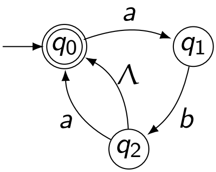


$x = aababa$





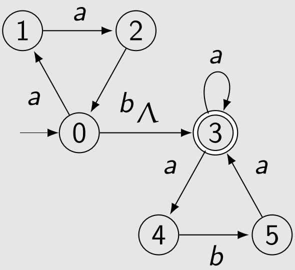
q_3
a
a
b
a
q_0
q_1
q_2
b
b
a
b
q_4
a, b



a
q_0
q_1
Λ
a
b
q_2

Example

$\{aab\}^*\{a, aba\}^*$



NFA

δ	Λ	a	b
0	$\{3\}$	$\{1\}$	$\varnothing$
3	$\varnothing$	$\{3, 4\}$	$\varnothing$

5-tuple $M = (Q, \Sigma, q_0, A, \delta)$

Definition ($\hookrightarrow$FA)

[deterministic] finite automaton
– $\delta : Q \times \Sigma \to Q$ *transition function*;

Definition (NFA)

nondeterministic finite automaton (with Λ-transitions)
– $\delta : Q \times (\Sigma \cup \{\Lambda\}) \to 2^Q$

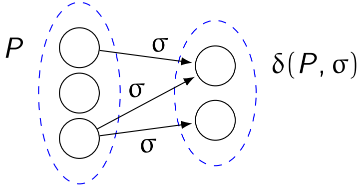
Where 2^Q the set of all subsets of Q:

$$2^Q = \{S \mid S \subseteq Q\}$$

Extended transition function without Λ-transitions

Extend δ to subsets P:

$\delta(P, \sigma) = \bigcup_{p \in P} \delta(p, \sigma) = \{q \in Q \mid q \in \delta(p, \sigma)$ for some $p \in P\}$.



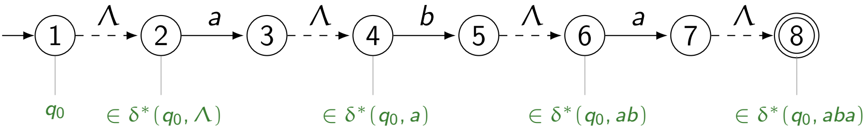
$\delta^*(q, \Lambda) = \{q\}$
$\delta^*(q, y\sigma) = \delta(\delta^*(q, y), \sigma)$

NFA $M = (Q, \Sigma, q_0, A, \delta)$ $\quad S \subseteq Q$

Definition

– $S \subseteq \Lambda(S)$

– $q \in \Lambda(S)$, then $\delta(q, \Lambda) \subseteq \Lambda(S)$

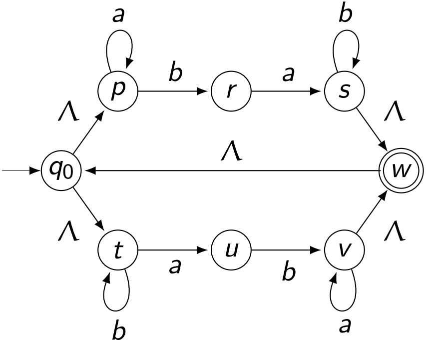


Definition

– $\delta^*(q, \Lambda) = \Lambda(\{q\})$ $\quad q \in Q$

– $\delta^*(q, y\sigma) = \Lambda(\ \delta(\ \delta^*(q, y), \sigma\ )\ )$ $\quad q \in Q,\ y \in \Sigma^*,\ \sigma \in \Sigma$

[M] D 3.13 & 3.14



Λ

$\Lambda(\{q_0\}) = \{q_0, p, t\}$

$\delta^*(q_0, \Lambda) = \{q_0, p, t\}$

$\Lambda \cdot a$

$\delta(\{q_0, p, t\}, a) = \{p, u\}$

$\delta^*(q_0, a) = \Lambda(\{p, u\}) = \{p, u\}$

$a \cdot b$

$\delta(\{p, u\}, b) = \{r, v\}$

$\delta^*(q_0, ab) = \Lambda(\{r, v\}) =$
$\{r, v, w, q_0, p, t\}$

$ab \cdot a$

$\delta(\{r, v, w, q_0, p, t\}, a) = \{s, v, p, u\}$

$\delta^*(q_0, aba) = \Lambda(\{s, v, p, u\}) =$
$\{s, w, q_0, p, t, v, u\}$

[M] E 3.15

NFA $M = (Q, \Sigma, q_0, A, \delta)$

Theorem

$q \in \delta^*(p, x)$ iff there is a path in [the transition graph of] M from p to q with label x (possibly including Λ-transitions).

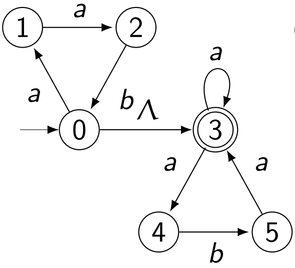
$\delta^*(q_0, x) = \varnothing$ no path for x from initial state

Definition

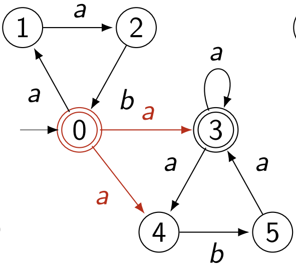
A string $x \in \Sigma^*$ is accepted by $M = (Q, \Sigma, q_0, A, \delta)$ if $\delta^*(q_0, x) \cap A \neq \varnothing$.
The *language $L(M)$ accepted* by M is the set of all strings accepted by M.

[M] D 3.14 [L] D 2.2

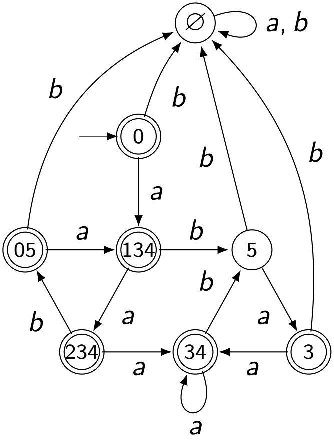
$L = \{aab\}^*\{a, aba\}^*$



NFA-Λ



NFA



FA

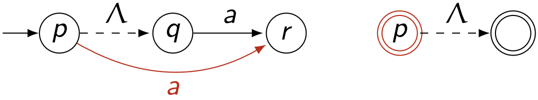
Theorem

For every language $L \subseteq \Sigma^*$ accepted by an NFA $M = (Q, \Sigma, q_0, A, \delta)$, there is an NFA M_1 with no Λ-transitions that also accepts L.

[M] T 3.17

The precise inductive proof of this result does not have to be known for the exam. However, the construction in the next slides has to be known.

Different from book!



Λ-removal

NFA $M = (Q, \Sigma, q_0, A, \delta)$

construct NFA $M_1 = (Q, \Sigma, q_0, A_1, \delta_1)$ without Λ-transitions

– whenever $q \in \Lambda_M(\{p\})$ and $r \in \delta(q, a)$, add r to $\delta_1(p, a)$

– whenever $\Lambda_M(\{p\}) \cap A \neq \varnothing$, add p to A_1.

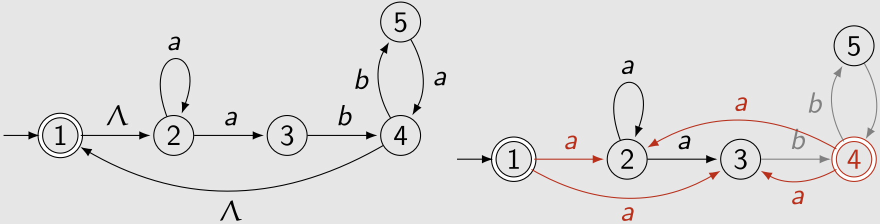
In particular,

– non-Λ-transitions are maintained

– $A \subseteq A_1$

Different from book!

Example

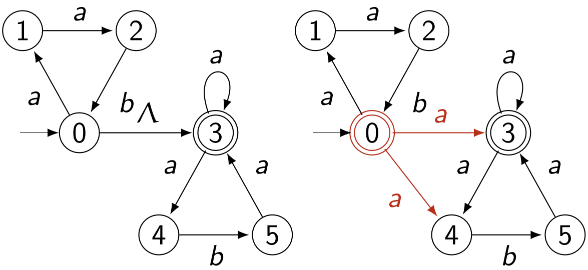


Construction book: $\delta_1(p, \sigma) = \delta^*(p, \sigma) \qquad \forall p \in Q, \forall \sigma \in \Sigma$

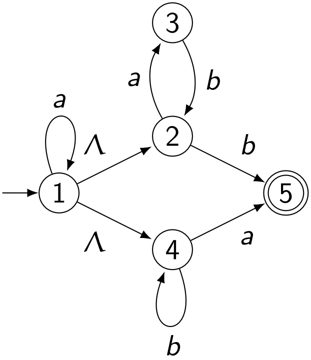
Accepting states: $A \cup \{q_0\}$ if $\Lambda \in L$, A otherwise

[M] E 3.19 but fewer edges!

$L = \{aab\}^*\{a, aba\}^*$



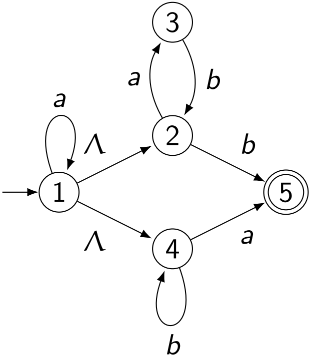
$\{a\}^*[\ \{ab\}^*\{b\} \cup \{b\}^*\{a\}\ ]$



q	$\delta(q,a)$	$\delta(q,b)$	$\delta(q,\Lambda)$	$\Lambda(\{q\})$
1	1	—	2, 4	1, 2, 4
2	3	5	—	2
3	—	2	—	3
4	5	4	—	4
5	—	—	—	5

[M] E 3.23

$\{a\}^*[\,\{ab\}^*\{b\} \cup \{b\}^*\{a\}\,]$



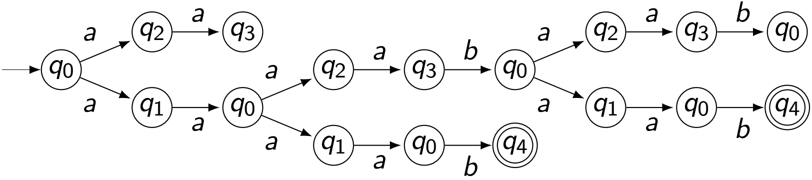
[M] E 3.23 but fewer edges!

Theorem

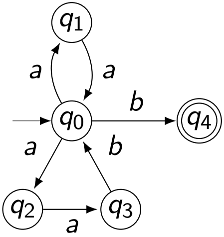
For every language $L \subseteq \Sigma^*$ accepted by an NFA $M = (Q, \Sigma, q_0, A, \delta)$ without Λ-transitions, there is an FA M_1 that also accepts L.

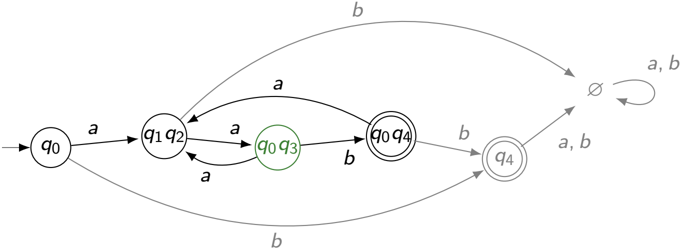
[M] T 3.18

The precise inductive proof of this result does not have to be known for the exam. However, the construction in the next slides has to be known.

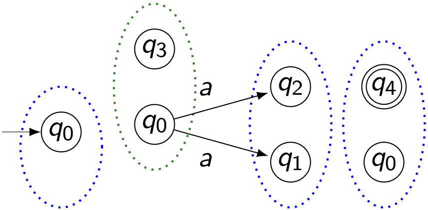


$\{q_0\}$ $\{q_1, q_2\}$ $\{q_0, q_3\}$ $\{q_1, q_2\}$ $\{q_0, q_3\}$ $\{q_0, q_4\}$ $\{q_1, q_2\}$ $\{q_0, q_3\}$ $\{q_0, q_4\}$





[M] E 3.6 and E 3.21



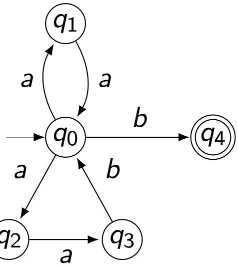
Subset construction

NFA $M = (Q, \Sigma, q_0, A, \delta)$ without Λ-transitions

construct FA $M_1 = (Q_1, \Sigma, \delta_1, q_1, A_1)$

- $Q_1 = 2^Q$, i.e., sets of states of M
- $q_1 = \{q_0\}$
- $A_1 = \{\ q \in Q_1 \mid q \cap A \neq \varnothing\ \}$
- $\delta_1(q, \sigma) = \bigcup_{p \in q} \delta(p, \sigma) \qquad \forall q \in Q_1, \forall \sigma \in \Sigma$

[M] Th 3.18

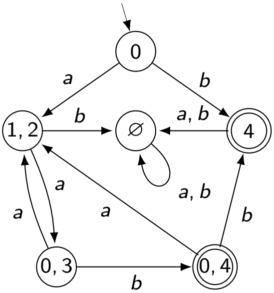


q	$\delta(q, a)$	$\delta(q, b)$
0	1, 2	4
1	0	—
2	3	—
3	—	0
4	—	—

[M] E 3.21. also ↪E 3.6

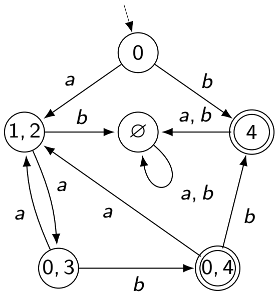
Once more $\{aa, aba\}^*\{b\}$

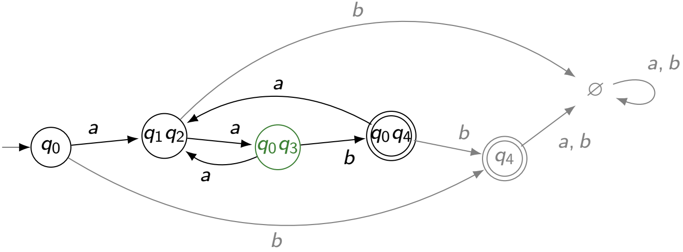
$L = \{aa, aab\}^*\{b\}$



Minimal this time, yet not always

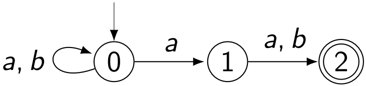
[M] E 3.21. also ↪E 3.6

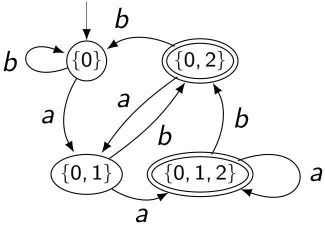


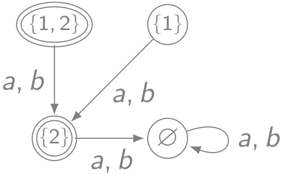


[M] E 3.6 and E 3.21

Example: *a* second symbol from end



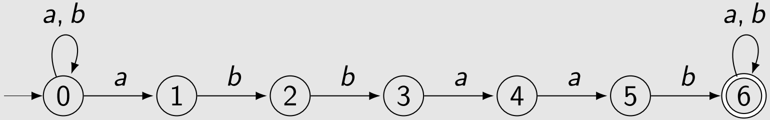




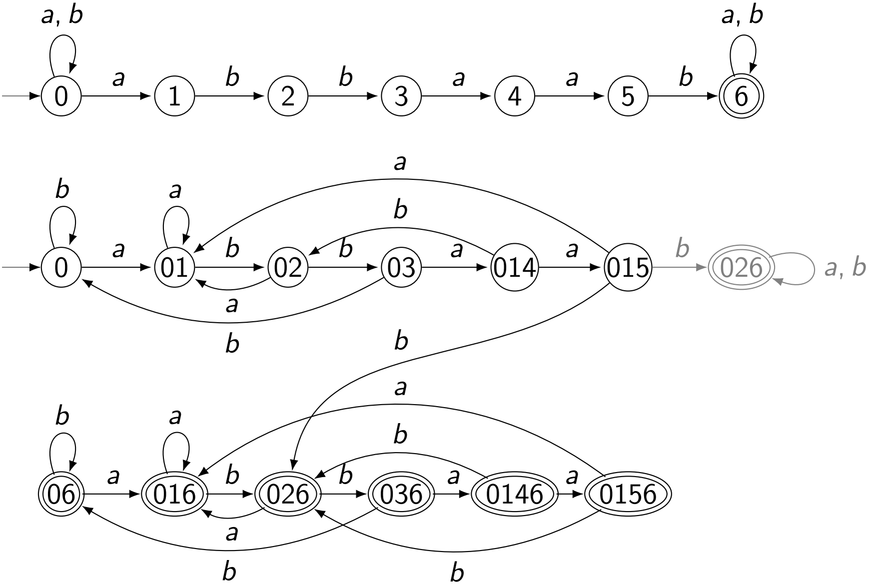
also ↪3rd from the end

Example

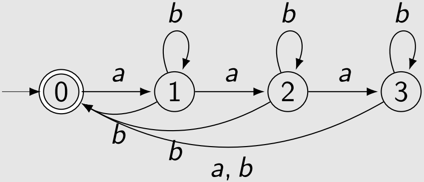
$L_3 = \{ x \in \{a, b\}^* \mid x \text{ contains the substring } abbaab \}$



[M] ↪E. 2.5 (deterministic)


a, b
a
b
b
a
a
b
a, b
0
1
2
3
4
5
6
b
a
a
b
b
a
a
b
a, b
0
01
02
03
014
015
026
a
b
b
b
a
b
a
b
b
a
a
06
016
026
036
0146
0156
a
b
b

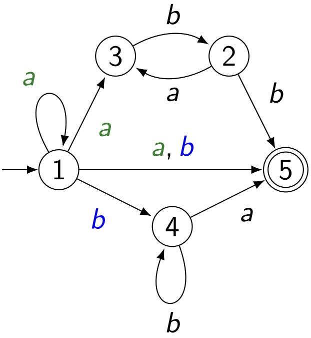
Example ($n = 4$)



All 2^n subsets are reachable, nonequivalent, states.

Example 3.23 revisited

$\{a\}^*[\,\{ab\}^*\{b\} \cup \{b\}^*\{a\}\,]$

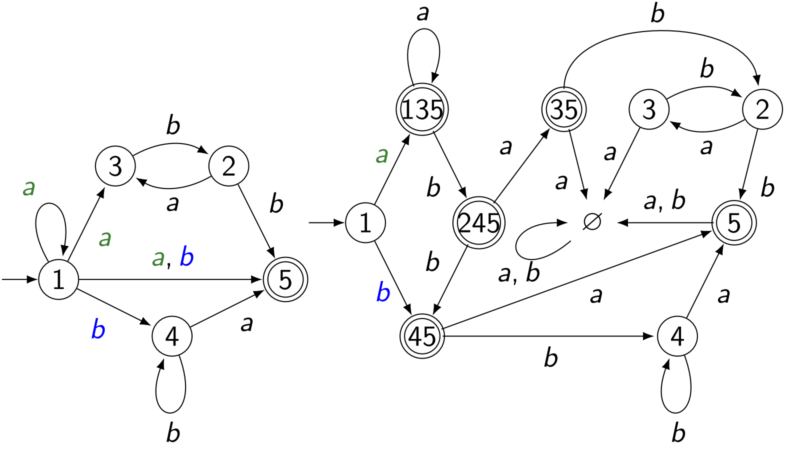


q	$\delta(q, a)$	$\delta(q, b)$
1	1, 3, 5	4, 5
2	3	5
3	—	2
4	5	4
5	—	—

[M] E 3.23 ctd.

Example 3.23 revisited

$\{a\}^*[\,\{ab\}^*\{b\} \cup \{b\}^*\{a\}\,]$



[M] E 3.23 ctd.

☒Example: Brzozowski minimization