Automata Theory

Mark van den Bergh / Rudy van Vliet

Bachelor Informatica
Data Science and Artificial Intelligence
Universiteit Leiden

Fall 2024

Universiteit
Leiden
Leiden Institute of
Advanced Computer Science

Let $L = \{x \in \{a, b\}^* \mid x \text{ does not contain } bb\}$.

The set $\{\Lambda, b, bb\}$ is pairwise L-distinguishable, because
$\Lambda b = b \in L$, but $bb \notin L$;
$\Lambda\Lambda = \Lambda \in L$, but $bb\Lambda = bb \notin L$;
$b\Lambda = b \in L$, but $bb\Lambda = bb \notin L$.

Or:
$L/\Lambda = L$;
$L/b = \{x \in L \mid x \text{ does not begin with } b\}$;
$L/bb = \emptyset$.
All are different.

Theorem

Suppose $M = (Q, \Sigma, q_0, A, \delta)$ is an FA accepting $L \subseteq \Sigma^*$.

If $x, y \in \Sigma^*$ are L-distinguishable, then $\delta^*(q_0, x) \neq \delta^*(q_0, y)$.

For every $n \geqslant 2$, if there is a set of n pairwise L-distinguishable strings in Σ^*, then Q must contain at least n states.

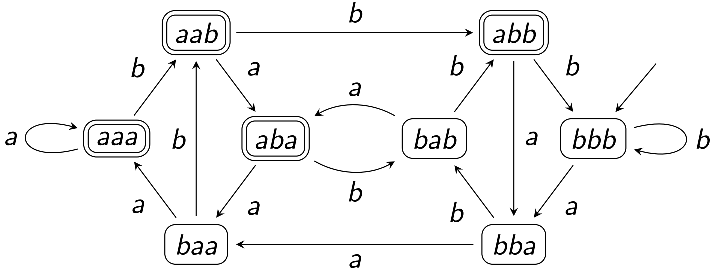
Hence, indeed: if $\delta^*(q_0, x) = \delta^*(q_0, y)$, then x and y are not L-distinguishable.

Proof. Suppose x and y are L-distinguishable. W.l.o.g. there exists some $z \in \Sigma^*$ such that $xz \in L$ and $yz \notin L$. In other words, $\delta^*(q_0, xz) \in A$ and $\delta^*(q_0, yz) \notin A$. Hence, $\delta^*(q_0, xz) \neq \delta^*(q_0, yz)$.
By Exercise 2.5, we may rewrite $\delta^*(q_0, xz) = \delta^*(\delta^*(q_0, x), z)$ and $\delta^*(q_0, yz) = \delta^*(\delta^*(q_0, y), z)$. Hence, we conclude that $\delta^*(\delta^*(q_0, x), z) \neq \delta^*(\delta^*(q_0, y), z)$, so also $\delta^*(q_0, x) \neq \delta^*(q_0, y)$ must hold.

[M] Thm 2.21

Strings with a in the 3rd symbol from the end

L the language of strings in $\{a, b\}^*$ with at least 3 symbols and an a in the 3rd position from the end.



[M] E. 2.24

Theorem

For every language $L \subseteq \Sigma^*$,
if there is an infinite set S of pairwise L-distinguishable strings,
then L cannot be accepted by a finite automaton.

[M] Thm 2.26

$L = \{a^i b^j c^j \mid i \geqslant 1 \text{ and } j \geqslant 0\} \cup \{b^j c^k \mid j \geqslant 0 \text{ and } k \geqslant 0\}$

We claim $\{ab^n \mid n \geqslant 1\}$ is pairwise L-distinguishable. Indeed, for $m \neq n$, we find that $ab^m c^m \in L$, but $ab^n c^m \neq L$.

[M] E 2.39

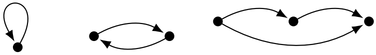
$Pal = \{x \in \{a, b\}^* \mid x = x^r\}$

We claim $\{a^n b \mid n \geqslant 1\}$ is pairwise L-distinguishable. Indeed, for $m \neq n$, we find that $a^m ba^m \in L$, but $a^n ba^m \notin L$.

[M] E. 2.27

R equivalence relation on X

- reflexive: $\forall x \in X : xRx$
- symmetric: $\forall x, y \in X : xRy \Leftrightarrow yRx$
- transitive: $\forall x, y, z \in X : xRy \wedge yRz \Rightarrow xRz$



equivalence class $[x]_R = \{ y \in X \mid yRx \}$
short: $[x]$
partition of X

[M] Sect. 1.3

Definition

For a language $L \subseteq \Sigma^*$, we define the relation $\equiv_L$ (an equivalence relation) on Σ^* as follows: for $x, y \in \Sigma^*$

$$x \equiv_L y \qquad \text{if and only if } x \text{ and } y \text{ are } L\text{-indistinguishable}$$

Check properties of equivalence relation!

Note: $x \equiv_L y$ if and only if $L/x = L/y$.

$\equiv_L$ is right invariant: $x \equiv_L y$ implies $xz \equiv_L yz$

Book uses I_L for $\equiv_L$

Example

$L = \{ x \in \{a, b\}^* \mid x \text{ ends with } aa \}$

Remember: $\{\Lambda, a, aa\}$ pairwise L-distinguishable.

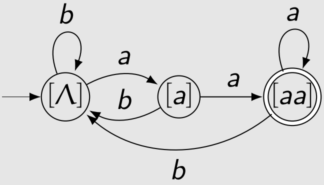
Equivalence classes:
$[\Lambda] = \{x \in \{a, b\}^* \mid x \text{ does not end in } a\}$;
$[a] = \{x \in \{a, b\}^* \mid x \text{ ends in } a \text{ but not in } aa\}$;
$[aa] = L$.

Note: $[\Lambda] \cup [a] \cup [aa] = \{a, b\}^*$.

From lecture 1:

Example

$L_1 = \{\, x \in \{a, b\}^* \mid x \text{ ends with } aa \,\}$



[M] E. 2.1

State q in FA $\approx$ $L_q = \{x \in \Sigma^* \mid \delta^*(q_0, x) = q\}$

Theorem

If $L \subseteq \Sigma^*$ can be accepted by a finite automaton, then the set Q_L of equivalence classes of the relation $\equiv_L$ is finite.

Conversely, if the set Q_L is finite,
the finite automaton $M_L = (Q_L, \Sigma, q_0, A, \delta)$ accepts L, where
$q_0 = [\Lambda]$
$A = \{q \in Q_L \mid q \subseteq L\}$
$\delta([x], \sigma) = [x\sigma]$

Finally, M_L has the fewest states of any FA accepting L.

Note:
If $x \in L$, then $[x] \subseteq L$ (L is union of equivalence classes)
Right invariant $x \equiv_L y$ implies $x\sigma \equiv_L y\sigma$

[M] Thm 2.36

Exercise 2.36.
For a certain language $L \subseteq \{a, b\}^*$, $\equiv_L$ has exactly four equivalence classes. They are $[\Lambda]$, $[a]$, $[ab]$ and $[b]$.

It is also true that the three strings a, aa, and abb are all equivalent, and that the two strings b and aba are equivalent.

Finally, $ab \in L$, but Λ and a are not in L, and b is not even a prefix of any element of L.

Draw an FA accepting L.

Example

Equivalence classes of $\equiv_L$, where $L = AnBn = \{a^n b^n \mid n \geqslant 0\}$

Note: $\{a^n \mid n \geqslant 1\}$ is pairwise distinguishable.

$[a^n] = \{a^n\}$, because $L/a^n = \{a^k b^{n+k} \mid k \geqslant 0\}$ all different.

Other classes:
$[ab] = L - \{\Lambda\}$;
$[b] = \{x \in \{a, b\}^* \mid xz \notin L \text{ for all } z \in \{a, b\}^*\}$;
...

Infinitely many equivalence classes, so no FA.

[M] E 2.37

Recall $L_q = \{ x \in \Sigma^* \mid \delta^*(q_0, x) = q \}$
Equivalence relation $\equiv_L$ induces equivalence relation $\equiv$ on states
Each L_q is subset of equivalence class under $\equiv_L$
L_p and L_q may be subset of same equivalence class, i.e., $L_p, L_q \subseteq [x]$ for some $x \in \Sigma^*$.
$p \equiv q \iff L_p$ and L_q are subset of same equivalence class
$p \not\equiv q \iff$ for some $z \in \Sigma^*$ exactly one of $\delta^*(p, z)$ and $\delta^*(q, z)$ is in A

Definition

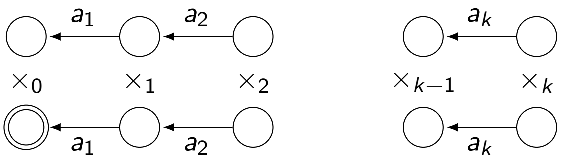
S_M: set of pairs (p, q) such that $p \not\equiv q$

1. If exactly one of p and q is in A, then $(p, q) \in S_M$
2. If for some $\sigma \in \Sigma$, $(\delta(p, \sigma), \delta(q, \sigma)) \in S_M$, then $(p, q) \in S_M$

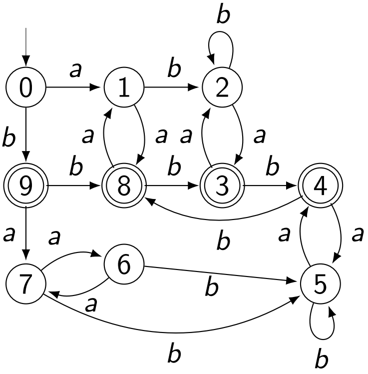
ALGORITHM mark pairs of non-equivalent states

start by marking pairs (p, q) where exactly one p, q in A

repeat
    for each unmarked pair (p, q)
        check whether there is a σ such that $(\,\delta(p,\sigma), \delta(q,\sigma)\,)$ is marked
        then mark (p, q)
until this pass does not mark new pairs

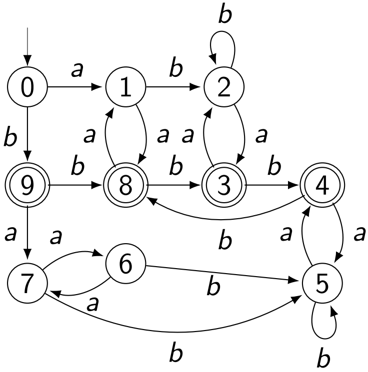


[M] Algo 2.40



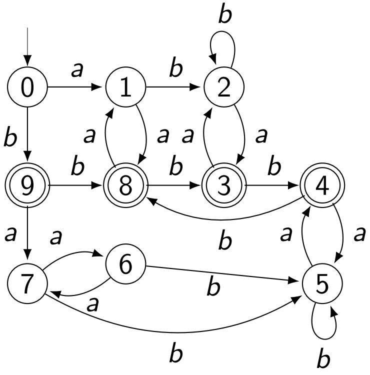
1	.								
2	.	.							
3	.	.	.						
4	.	.	.	.					
5	.	.	.	.	.				
6	.	.	.	.	.	.			
7	.	.	.	.	.	.	.		
8	.	.	.	.	.	.	.	.	
9	.	.	.	.	.	.	.	.	.
	0	1	2	3	4	5	6	7	8

[M] Fig 2.42



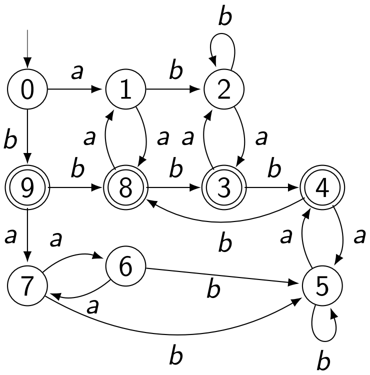
	0	1	2	3	4	5	6	7	8
1	·								
2	·	·							
3	1	1	1						
4	1	1	1	·					
5	·	·	·	1	1				
6	·	·	·	1	1	·			
7	·	·	·	1	1	·	·		
8	1	1	1	·	·	1	1	1	
9	1	1	1	·	·	1	1	1	·

[M] Fig 2.42



1	2								
2	2	·							
3	1	1	1						
4	1	1	1	·					
5	2	·	·	1	1				
6	2	2	2	1	1	2			
7	2	2	2	1	1	2	·		
8	1	1	1	·	·	1	1	1	
9	1	1	1	2	·	1	1	1	2
	0	1	2	3	4	5	6	7	8

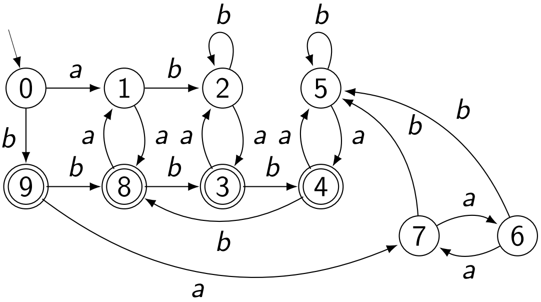
[M] Fig 2.42

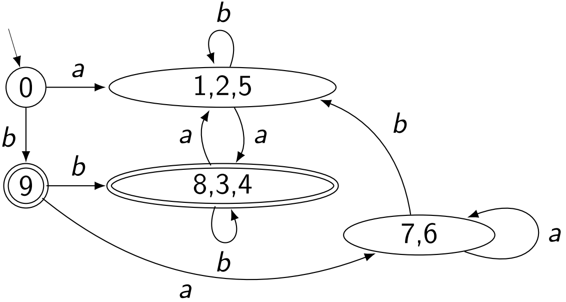


1	2								
2	2	·							
3	1	1	1						
4	1	1	1	·					
5	2	·	·	1	1				
6	2	2	2	1	1	2			
7	2	2	2	1	1	2	·		
8	1	1	1	·	·	1	1	1	
9	1	1	1	2	3	1	1	1	2
	0	1	2	3	4	5	6	7	8

Resulting (minimal) FA...

[M] Fig 2.42





[M] Fig 2.42