Automata Theory

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Definition (FA)

[deterministic] finite automaton 5-tuple $M = (Q, \Sigma, q_0, A, \delta)$,

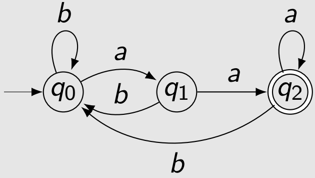
- Q finite set *states*;
- Σ finite *input alphabet*;
- $q_0 \in Q$ *initial* state;
- $A \subseteq Q$ *accepting* states;
- $\delta : Q \times \Sigma \to Q$ *transition function*.

[M] D 2.11 Finite automaton

[L] D 2.1 Deterministic finite accepter, has 'final' states

Example

$L_1 = \{ x \in \{a, b\}^* \mid x \text{ ends with } aa \}$



$Q = \{q_0, q_1, q_2\}$
$\Sigma = \{a, b\}$
q_0 initial state
$A = \{q_2\}$

δ	a	b
q_0	q_1	q_0
q_1	q_2	q_0
q_2	q_2	q_0

[M] E. 2.1

FA $M = (Q, \Sigma, q_0, A, \delta)$

Definition

extended transition function $\delta^* : Q \times \Sigma^* \to Q$, such that

– $\delta^*(q, \Lambda) = q$ for $q \in Q$

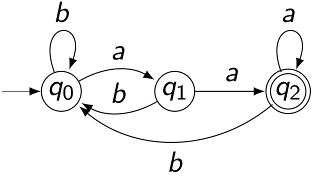
– $\delta^*(q, y\sigma) = \delta(\ \delta^*(q, y), \sigma\ )$ for $q \in Q, y \in \Sigma^*, \sigma \in \Sigma$

[M] D 2.12 [L] p.40/1

Theorem

$q = \delta^*(p, w)$ iff there is a path in [the transition graph of] M from p to q with label w.

[L] Th 2.1



$$q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_2 \xrightarrow{b} q_0 \xrightarrow{b} q_0$$

$\delta^*(q_0, \Lambda)$ $\delta^*(q_0, a)$ $\delta^*(q_0, aa)$ $\delta^*(q_0, aab)$ $\delta^*(q_0, aabb)$

$\delta^*(q_0, aabb) = q_0$:

$$\delta^*(q_0, \Lambda) = q_0$$

$$\delta^*(q_0, a) = \delta^*(q_0, \Lambda\, a) = \delta(\delta^*(q_0, \Lambda), a) = \delta(q_0, a) = q_1$$

$$\delta^*(q_0, aa) = \delta(\delta^*(q_0, a), a) = \delta(q_1, a) = q_2$$

$$\delta^*(q_0, aab) = \delta(\delta^*(q_0, aa), b) = \delta(q_2, b) = q_0$$

$$\delta^*(q_0, aabb) = \delta(\delta^*(q_0, aab), b) = \delta(q_0, b) = q_0$$

Definition

Let $M = (Q, \Sigma, q_0, A, \delta)$ be an FA, and let $x \in \Sigma^*$. The string x is *accepted* by M if $\delta^*(q_0, x) \in A$.
The *language accepted* by $M = (Q, \Sigma, q_0, A, \delta)$ is the set $L(M) = \{\, x \in \Sigma^* \mid x \text{ is accepted by } M \,\}$

[M] D 2.14 [L] D 2.2

Construction

FA $M = (Q, \Sigma, q_0, A, \delta)$,

let $M^c = (Q, \Sigma, q_0, Q - A, \delta)$

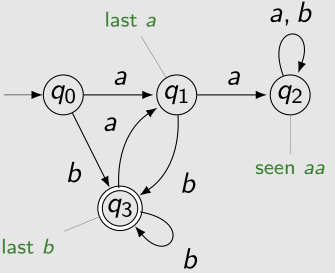
Theorem

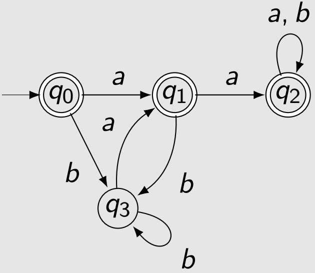
$L(M^c) = \Sigma^* - L(M)$

Proof. Suppose $x \in L(M^c)$. Then x is accepted by M^c, so it holds that $\delta^*(q_0, x) \in Q - A$. Hence, x is not accepted by M, so $x \notin L(M)$, so $x \in \Sigma^* - L(M)$.
Suppose $x \in \Sigma^* - L(M)$. Then $x \notin L(M)$, so x is not accepted by M. Hence, $\delta^*(q_0, x) \notin A$, so $\delta^*(q_0, x) \in Q - A$, so x is accepted by M^c, that is, $x \in L(M^c)$.

Example

$L = \{ x \in \{a, b\}^* \mid x \text{ ends with } b \text{ and does not contain } aa \}$





$\neg(P \wedge Q) = \neg P \vee \neg Q$

$L^c = \{ x \in \{a, b\}^* \mid x \text{ does not end with } b \text{ or contains } aa \}$

FA $M_i = (Q_i, \Sigma, q_i, A_i, \delta_i) \quad i = 1, 2$

Product construction

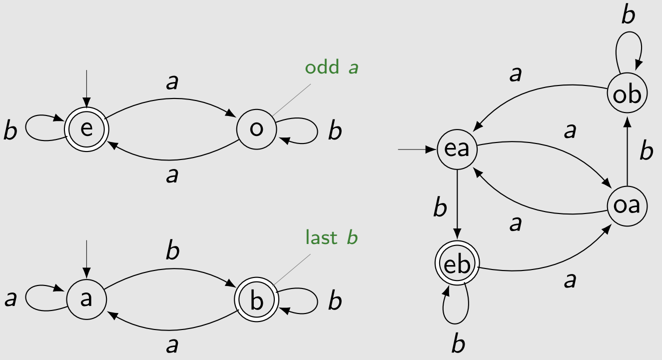
construct FA $M = (Q, \Sigma, q_0, A, \delta)$ such that
– $Q = Q_1 \times Q_2$
– $q_0 = (q_1, q_2)$
– $\delta(\ (p, q), \sigma) = (\ \delta_1(p, \sigma), \delta_2(q, \sigma)\ )$
– A as needed

Theorem (2.15 Parallel simulation)

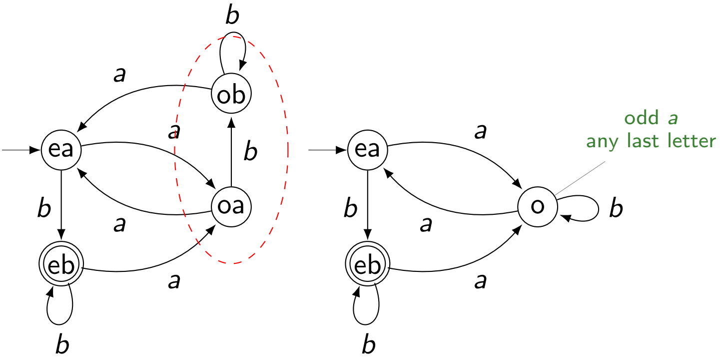
– $A = \{(p, q) \mid p \in A_1$ *or* $q \in A_2\}$, *then* $L(M) = L(M_1) \cup L(M_2)$
– $A = \{(p, q) \mid p \in A_1$ *and* $q \in A_2\}$, *then* $L(M) = L(M_1) \cap L(M_2)$
– $A = \{(p, q) \mid p \in A_1$ *and* $q \notin A_2\}$, *then* $L(M) = L(M_1) - L(M_2)$

[M] Sect 2.2

Example (Even number of a, and ending with b)



Even number of *a* and ending with *b*



Exercise 2.11.
Use induction to show that for every $x \in \Sigma^*$ and every $(p, q) \in Q$, $\delta^*((p, q), x) = (\delta_1^*(p, x), \delta_2^*(q, x))$

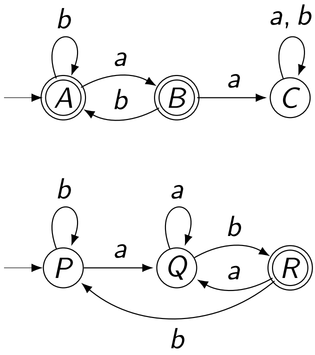
Proof. It clearly holds that

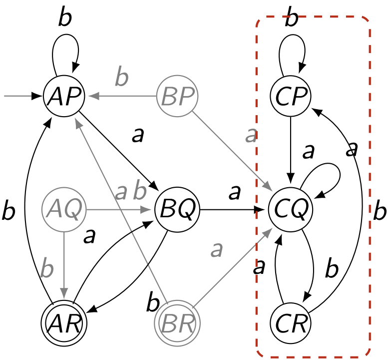
$$\delta^*((p, q), \Lambda) = (p, q) = (\delta_1^*(p, \Lambda), \delta_2^*(q, \Lambda)).$$

Next, suppose the statement holds for all x with $|x| < n$ and consider some $x = y\sigma$ of length n. Then

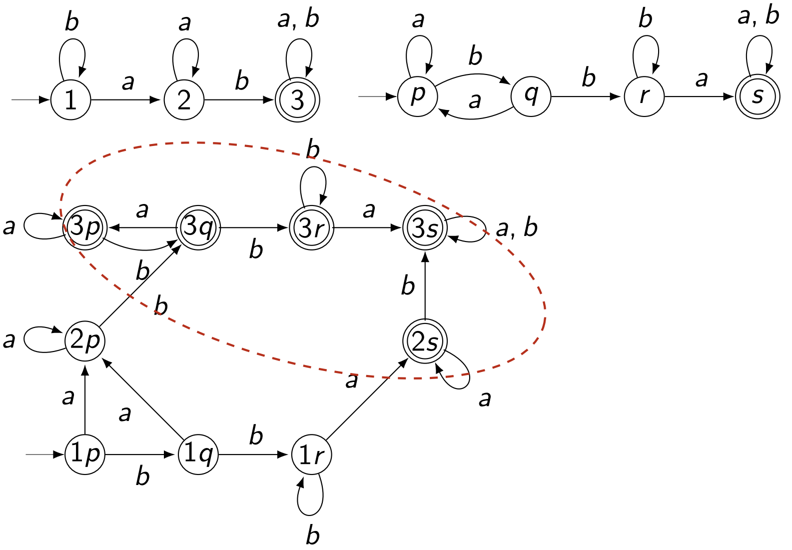
$$\begin{aligned}
\delta^*((p, q), x) &= \delta(\delta^*((p, q), y), \sigma) && \text{defn. } \delta^*, x = y\sigma \\
&= \delta((\delta_1^*(p, y), \delta_2^*(q, y)), \sigma) && \text{ind. hypothesis} \\
&= (\delta_1(\delta_1^*(p, y), \sigma), \delta_2(\delta_2^*(q, y), \sigma)) && \text{defn. product FA} \\
&= (\delta_1^*(p, y\sigma), \delta_2^*(q, y\sigma)) && \text{defn. } \delta_1^*, \delta_2^* \\
&= (\delta_1^*(p, x), \delta_2^*(p, x)). && x = y\sigma
\end{aligned}$$

Example: intersection 'and' (product construction)



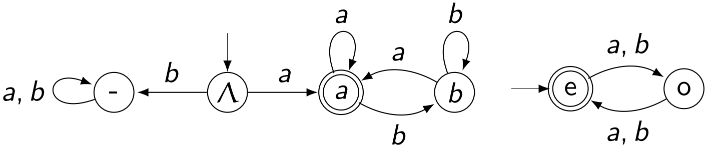


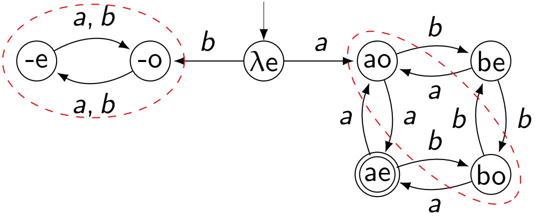
[M] E 2.16



[M] E. 2.18, see also ↪subset construction

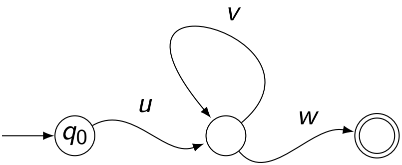
$L = \{ \ w \in \{a, b\}^* \mid w \text{ starts and ends with an } a, \text{ and } |w| \text{ is even } \}$





Theorem

REG is closed under complement, union and intersection.



[M] Fig. 2.28

Regular language is language accepted by an FA.

Theorem

Suppose L is a language over the alphabet Σ. If L is accepted by a finite automaton M, and if n is the number of states of M, then

$\forall$ *for every $x \in L$*
satisfying $|x| \geqslant n$

$\exists$ *there are three strings u, v, and w,*
such that $x = uvw$ and the following three conditions are true:
(1) $|uv| \leqslant n$,
(2) $|v| \geqslant 1$

$\forall$ *and (3) for all $m \geqslant 0$, $uv^m w$ belongs to L*

[M] Thm. 2.29

In other words:

Theorem

$\forall$ *For every regular language* L

$\exists$ *there exists a constant* $n \geqslant 1$

such that

$\forall$ *for every* $x \in L$

with $|x| \geqslant n$

$\exists$ *there exists a decomposition* $x = uvw$

with (1) $|uv| \leqslant n$,

and (2) $|v| \geqslant 1$

such that

$\forall$ *(3) for all* $m \geqslant 0$, $uv^m w \in L$

if $L = L(M)$ then $n = |Q|$.

[M] Thm. 2.29

In other words:

Theorem

If L *is a regular language, then*

$\exists$ *there exists a constant* $n \geqslant 1$

such that

$\forall$ *for every* $x \in L$

with $|x| \geqslant n$

$\exists$ *there exists a decomposition* $x = uvw$

with (1) $|uv| \leqslant n$,

and (2) $|v| \geqslant 1$

such that

$\forall$ *(3) for all* $m \geqslant 0$, $uv^m w \in L$

if $L = L(M)$ then $n = |Q|$.

Introduction to Logic: $p \to q \iff \neg q \to \neg p$

Theorem

If

$\forall$ *for every* $n \geqslant 1$

$\exists$ *there exists* $x \in L$

with $|x| \geqslant n$

such that

$\forall$ *for every decomposition* $x = uvw$

with (1) $|uv| \leqslant n$,

and (2) $|v| \geqslant 1$

$\exists$ *(3) there exists* $m \geqslant 0$,

such that

$uv^m w \notin L$

then L *is not a regular language.*

[M] Thm. 2.29

Example

$L = AnBn = \{a^n b^n \mid n \geqslant 0\}$ is not accepted by FA.

[M] E 2.30

Proof: by contradiction. Assume that L is accepted by FA with n states.
Take $x = a^n b^n$. Then $x \in L$, and $|x| = 2n \geqslant n$.
Thus there exists a decomposition $x = uvw$ such that $|uv| \leqslant n$ with v nonempty, and $uv^m w \in L$ for every m.
Whatever this decomposition is, v consists of a's only. Consider $m = 0$.
Deleting v from the string x will delete a number of a's. So $uv^0 w$ is of the form $a^{n'} b^n$ with $n' < n$.
This string is not in L; a contradiction. ($m \geqslant 2$ would also yield contradiction)
So, L is not regular.

Example

$L = AeqB = \{x \in \{a, b\}^* \mid \; n_a(x) = n_b(x) \; \}$ is not accepted by FA.

[M] E 2.30

Exactly the same argument can be used (verbatim) to prove that $L =$ AeqB is not regular.

We can also apply closure properties of REG to see that AeqB is not regular, as follows.

Assume AeqB is regular. Then also AnBn $=$ AeqB $\cap\, a^* b^*$ is regular, as regular languages are closed under intersection.
This is a contradiction, as we just have argued that AnBn is not regular.
Thus, also AeqB is not regular.