From lecture 3:

Definition

Let L be language over Σ, and let $x, y \in \Sigma^*$.
Then x, y are *distinguishable* wrt L (*L-distinguishable*),
if there exists $z \in \Sigma^*$ with

$$xz \in L \text{ and } yz \notin L \quad \text{or} \quad xz \notin L \text{ and } yz \in L$$

Such z *distinguishes* x and y wrt L.

Equivalent definition:
let $L/x = \{ z \in \Sigma^* \mid xz \in L \}$

x and y are L-*distinguishable* if $L/x \neq L/y$.
Otherwise, they are L-*indistinguishable*.

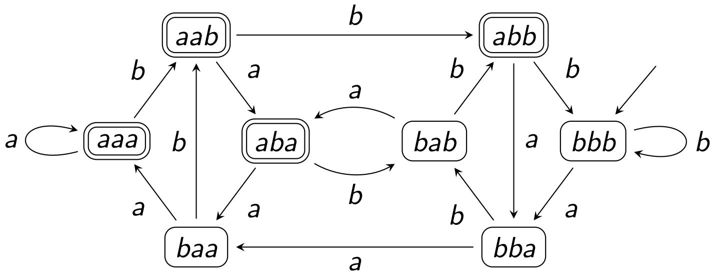
The strings in a set $S \subseteq \Sigma^*$ are *pairwise L-distinguishable*, if for every pair x, y of distinct strings in S, x and y are L-distinguishable.

Definition independent of FAs

[M] D 2.20

Strings with a in the nth symbol from the end

[M] E. 2.24



[M] E. 2.24

From lecture 3:

Theorem

Suppose $M = (Q, \Sigma, q_0, A, \delta)$ is an FA accepting $L \subseteq \Sigma^*$.

If $x, y \in \Sigma^*$ are L-distinguishable, then $\delta^*(q_0, x) \neq \delta^*(q_0, y)$.

For every $n \geqslant 2$, if there is a set of n pairwise L-distinguishable strings in Σ^*, then Q must contain at least n states.

Hence, indeed: if $\delta^*(q_0, x) = \delta^*(q_0, y)$, then x and y are not L-distinguishable.

Proof. . .

[M] Thm 2.21

Theorem

For every language $L \subseteq \Sigma^*$,
if there is an infinite set S of pairwise L-distinguishable strings,
then L cannot be accepted by a finite automaton.

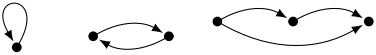
Proof...

[M] Thm 2.26

$Pal = \{x \in \{a, b\}^* \mid x = x^r\}$

[M] E. 2.27

R equivalence relation on A
– reflexive xRx for all ...
– symmetric xRy then yRx
– transitive xRy and yRz then xRz



equivalence class $[x]_R = \{ y \in A \mid yRx \}$
short: $[x]$
partition A

[M] Sect. 1.3

Definition

For a language $L \subseteq \Sigma^*$, we define the relation $\equiv_L$ (an equivalence relation) on Σ^* as follows: for $x, y \in \Sigma^*$

$$x \equiv_L y \qquad \text{if and only if } x \text{ and } y \text{ are } L\text{-indistinguishable}$$

Equivalence relation...

right invariant $\quad x \equiv_L y$ implies $xz_1 \equiv_L yz_1$

Book uses I_L for $\equiv_L$

Example

$L_1 = \{ x \in \{a, b\}^* \mid x \text{ ends with } aa \}$

L/x for $x = \Lambda, a, b, aa \ldots$
$L/\Lambda = \text{L}$
$L/a = \{a\} \cup L$
$L/b = L$
$L/aa = \{\Lambda, a\} \cup L$

Equivalence classes / partitioning of
$\{a, b\}^* = \{\Lambda, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, \ldots\} \ldots$

Example

Equivalence classes of $\equiv_L$, where $L = AnBn = \{a^i b^i \mid i \geqslant 0\}$

[M] E 2.37

Example

Equivalence classes of $\equiv_L$, where $L = AnBn = \{a^i b^i \mid i \geqslant 0\}$

$\{\Lambda\}$, $\{a\}$, $\{a^2\}$, $\{a^3\}$, ...
$\{a^i b^i \mid i \geqslant 1\}$
$\{a^{i+1} b^i \mid i \geqslant 1\}$, $\{a^{i+2} b^i \mid i \geqslant 1\}$, $\{a^{i+3} b^i \mid i \geqslant 1\}$, ...
$\{x \in \{a, b\}^* \mid x$ is not a prefix of any element of $L\}$
$= \{b, ba, bb, aba, abb, baa, \ldots\}$

Infinitely many equivalence classes

quotients

- $L/a^i = \{ a^k b^{i+k} \mid k \geqslant 0 \}$
- $L/a^{i+k} b^i = \{ b^k \}$ $\quad i > 0, k \geqslant 0$
- $L/a^i b^j = L/xbay = \varnothing$ $\quad j > i$

[M] E 2.37

Example

$L_1 = \{ x \in \{a, b\}^* \mid x \text{ ends with } aa \}$

L/x for $x = \Lambda, a, b, aa \ldots$

Equivalence classes / partitioning of
$\{a, b\}^* = \{\Lambda, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, \ldots\}$:

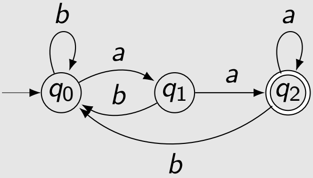
$\{\Lambda, b, ab, bb, aab, abb, \ldots\}$
$\{a, ba, aba, \ldots\}$
$\{aa, aaa, baa, \ldots\}$

Finitely many equivalence classes

From lecture 1:

Example

$L_1 = \{ x \in \{a, b\}^* \mid x \text{ ends with } aa \}$



[M] E. 2.1

State q in FA $\approx$ $L_q = \{x \in \Sigma^* \mid \delta^*(q_0, x) = q\}$

From lecture 3:

Theorem

Suppose $M = (Q, \Sigma, q_0, A, \delta)$ is an FA accepting $L \subseteq \Sigma^*$.

If $x, y \in \Sigma^*$ are L-distinguishable, then $\delta^*(q_0, x) \neq \delta^*(q_0, y)$.

For every $n \geqslant 2$, if there is a set of n pairwise L-distinguishable strings in Σ^*, then Q must contain at least n states.

Proof. . .

[M] Thm 2.21

In other words: if $\delta^*(q_0, x) = \delta^*(q_0, y)$, then x, y are L-indistinguishable.
Each L_q is subset of equivalence class

Theorem

If $L \subseteq \Sigma^*$ can be accepted by a finite automaton, then the set Q_L of equivalence classes of the relation $\equiv_L$ is finite.

Conversely, if the set Q_L is finite,
the finite automaton $M_L = (Q_L, \Sigma, q_0, A, \delta)$ accepts L, where
$q_0 = \ldots$
$A = \ldots$
$\delta([x], \sigma) = \ldots$

[M] Thm 2.36

Theorem

If $L \subseteq \Sigma^*$ can be accepted by a finite automaton, then the set Q_L of equivalence classes of the relation $\equiv_L$ is finite.

Conversely, if the set Q_L is finite,
the finite automaton $M_L = (Q_L, \Sigma, q_0, A, \delta)$ accepts L, where
$q_0 = [\Lambda]$
$A = \{q \in Q_L \mid q \subseteq L\}$
$\delta([x], \sigma) = [x\sigma]$

Finally, M_L has the fewest states of any FA accepting L.

Note:
If $x \in L$, then $[x] \subseteq L$ (L is union of equivalence classes)
Right invariant $x \equiv_L y$ implies $x\sigma \equiv_L y\sigma$

[M] Thm 2.36

From lecture 3:

Theorem

For every language $L \subseteq \Sigma^*$,
if there is an infinite set S of pairwise L-distinguishable strings,
then L cannot be accepted by a finite automaton.

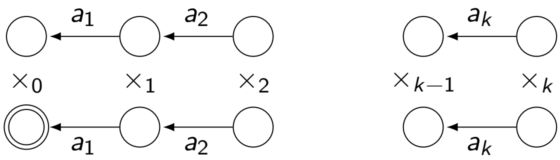
Proof. . .

[M] Thm 2.26

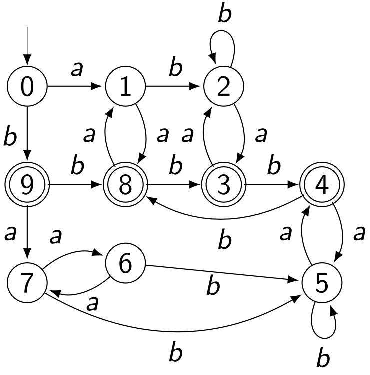
ALGORITHM mark pairs of non-equivalent states

start by marking pairs (p, q) where exactly one p, q in A

repeat
    for each unmarked pair (p, q)
        check whether there is a σ such that $(\, \delta(p, \sigma), \delta(p, \sigma)\,)$ is marked
        then mark (p, q)
until this pass does not mark new pairs



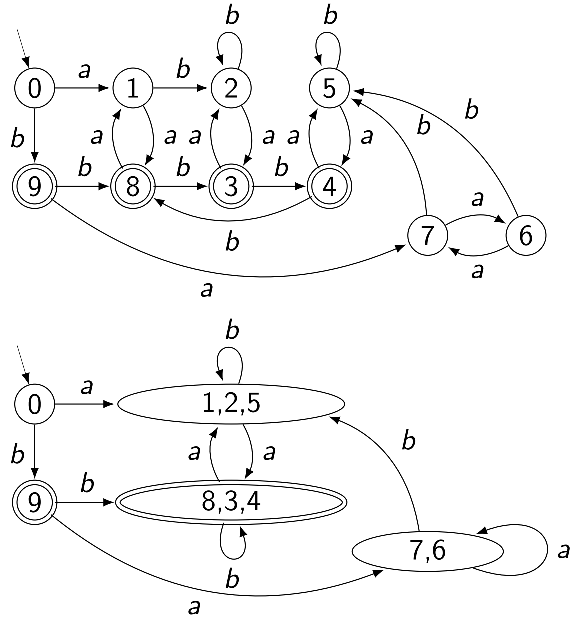
[M] Algo 2.40



	0	1	2	3	4	5	6	7	8
1	2								
2	2	.							
3	1	1	1						
4	1	1	1	.					
5	2	.	.	1	1				
6	2	2	2	1	1	2			
7	2	2	2	1	1	2	.		
8	1	1	1	.	.	1	1	1	
9	1	1	1	2	3	1	1	1	2

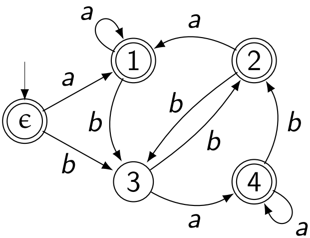
Resulting (minimal) FA. . .

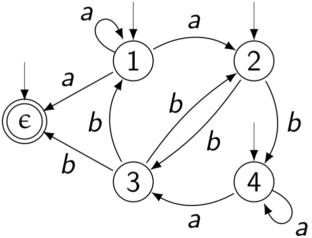
[M] Fig 2.42

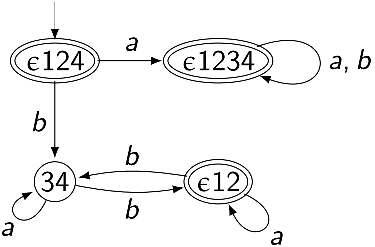


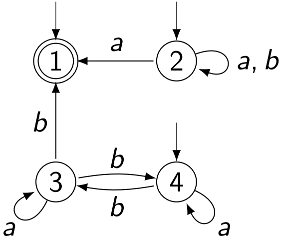
[M] Fig 2.42

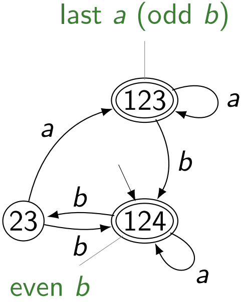
Example: Brzozowski minimization











ABOVE

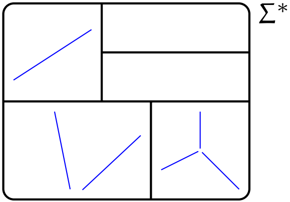
Brzozowski observes that one can minimize an FA by performing the following operations twice: invert (mirror), then determinize.
It is rather magical that this indeed works.
The method is in theory rather unfavourable, because of the exponentiation when detrminizing, but in practice seems not too slow.

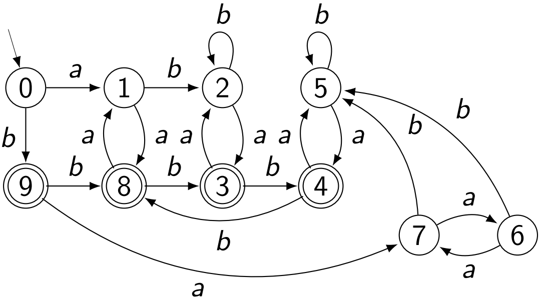
$L = L(M)$

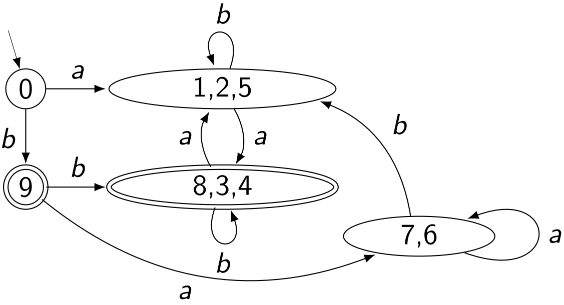
$\equiv_M$ state $\delta^*(q_0, x)$

$\equiv_L$ "future" L/x

$x \equiv_M y$, then $x \equiv_L y$.



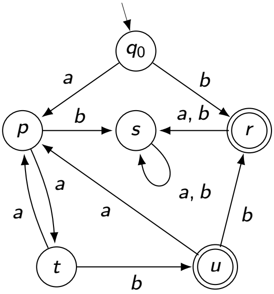




[M] Fig 2.42

From lecture 3:

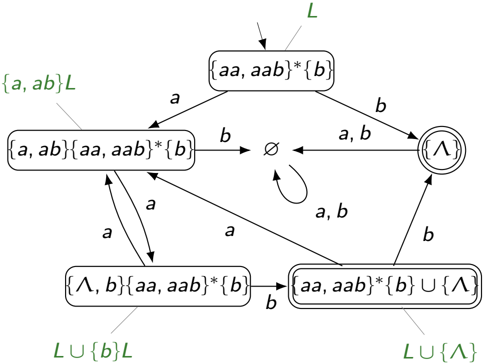
$L = \{aa, aab\}^*\{b\}$

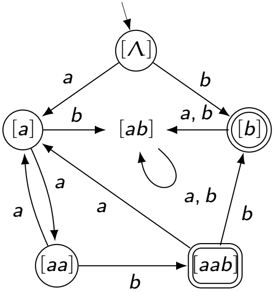


[M] E 2.22

$L = \{aa, aab\}^*\{b\}$

$L/\sigma = \{\, z \in \Sigma^* \mid \sigma z \in L \,\}$ $\quad L \xrightarrow{\sigma} L/\sigma$ $\qquad [x] \xrightarrow{\sigma} [x\sigma]$





[M] E 2.22 see ↪E 3.6