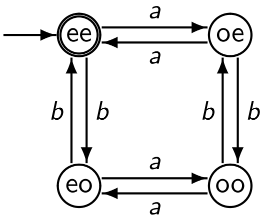
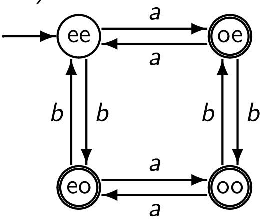
**2.1(g)** All strings over $\{a, b\}$ in which both the number of $a$'s and the number of $b$'s is even.




**2.1(g2)** All strings over $\{a, b\}$ in which either the number of $a$'s or the number of $b$'s is odd (or both).




[M] Exercise 2.1(g)

## Definition (FA)

*[deterministic] finite automaton*   5-tuple   $M = (Q, \Sigma, q_0, A, \delta)$,

– $Q$   finite set   *states*;

– $\Sigma$   finite *input alphabet*;

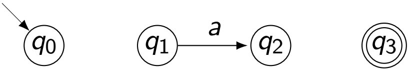
– $q_0 \in Q$   *initial* state;

– $A \subseteq Q$   *accepting* states;

– $\delta : Q \times \Sigma \to Q$   *transition function*.

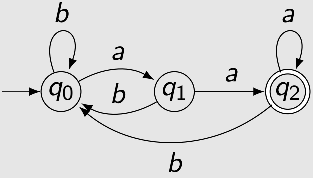
[M] D 2.11 Finite automaton

[L] D 2.1 Deterministic finite accepter, has 'final' states




## Example

$L_1 = \{ x \in \{a, b\}^* \mid x \text{ ends with } aa \}$




| $\delta$ | $a$ | $b$ |
|---|---|---|
| $q_0$ | $q_1$ | $q_0$ |
| $q_1$ | $q_2$ | $q_0$ |
| $q_2$ | $q_2$ | $q_0$ |

[M] E. 2.1

## Definition (FA)

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– $A \subseteq Q$ *accepting* states;

– $\delta : Q \times \Sigma \to Q$ *transition function*.

[M] D 2.11 Finite automaton

[L] D 2.1 Deterministic finite accepter, has 'final' states

FA $M = (Q, \Sigma, q_0, A, \delta)$

## Definition

*extended transition function* $\delta^* : Q \times \Sigma^* \to Q$, such that

– $\delta^*(q, \Lambda) = q$ for $q \in Q$

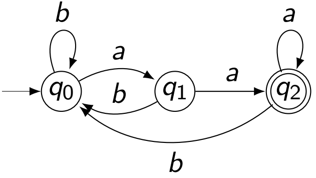
– $\delta^*(q, y\sigma) = \delta(\ \delta^*(q, y), \sigma\ )$ for $q \in Q, y \in \Sigma^*, \sigma \in \Sigma$

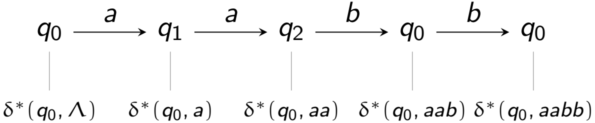
[M] D 2.12 [L] p.40/1

## Theorem

*$q = \delta^*(p, w)$ iff there is a path in [the transition graph of] $M$ from $p$ to $q$ with label $w$.*

[L] Th 2.1




$\delta^*(q_0, aabb) = q_0$ :

$$\delta^*(q_0, \Lambda) = q_0$$

$$\delta^*(q_0, a) = \delta^*(q_0, \Lambda\, a) = \delta(\delta^*(q_0, \Lambda), a) = \delta(q_0, a) = q_1$$

$$\delta^*(q_0, aa) = \delta(\delta^*(q_0, a), a) = \delta(q_1, a) = q_2$$

$$\delta^*(q_0, aab) = \delta(\delta^*(q_0, aa), b) = \delta(q_2, b) = q_0$$

$$\delta^*(q_0, aabb) = \delta(\delta^*(q_0, aab), b) = \delta(q_0, b) = q_0$$

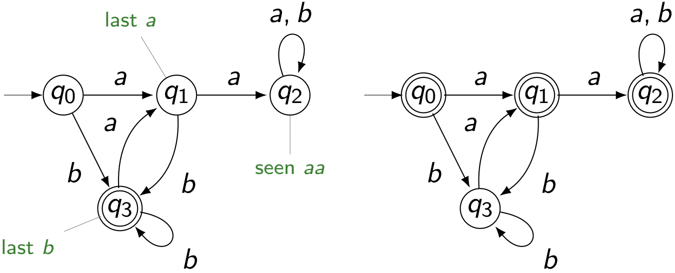
## Definition

Let $M = (Q, \Sigma, q_0, A, \delta)$ be an FA, and let $x \in \Sigma^*$. The string $x$ is *accepted* by $M$ if $\delta^*(q_0, x) \in A$.
The *language accepted* by $M = (Q, \Sigma, q_0, A, \delta)$ is the set
$L(M) = \{ x \in \Sigma^* \mid x \text{ is accepted by } M \}$

[M] D 2.14 [L] D 2.2

*From lecture 1:*

$L_2 = \{ x \in \{a, b\}^* \mid x \text{ ends with } b \text{ and does not contain } aa \}$




$\neg(P \wedge Q) = \neg P \vee \neg Q$

$L_2^c = \{ x \in \{a, b\}^* \mid x \text{ does not end with } b \text{ or contains } aa \}$

# Complement, construction

**Construction**

FA $M = (Q, \Sigma, q_0, A, \delta)$,

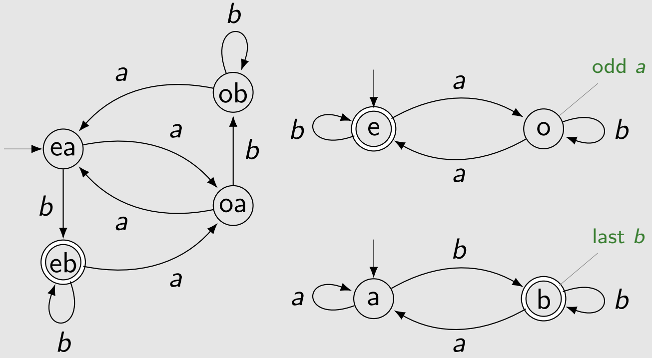
let $M^c = (Q, \Sigma, q_0, Q - A, \delta)$

**Theorem**

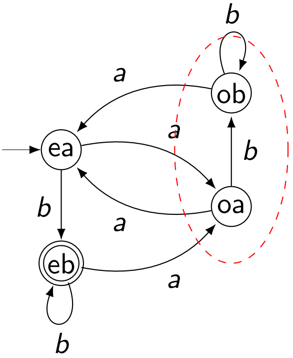
$L(M^c) = \Sigma^* - L(M)$

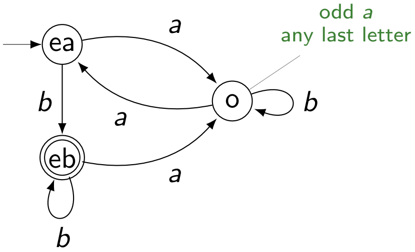
Proof...

## Example (Even number of *a*, and ending with *b*)



Even number of *a* and ending with *b*



FA $M_i = (Q_i, \Sigma, q_i, A_i, \delta_i) \quad i = 1, 2$

## Product construction

construct FA $M = (Q, \Sigma, q_0, A, \delta)$ such that
- $Q = Q_1 \times Q_2$
- $q_0 = (q_1, q_2)$
- $\delta(\ (p, q), \sigma) = (\ \delta_1(p, \sigma), \delta_2(q, \sigma)\ )$
- $A$ as needed

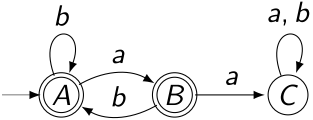
## Theorem (2.15 Parallel simulation)

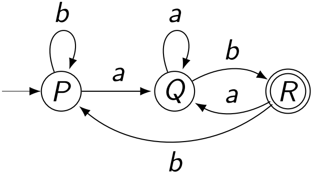
- $A = \{(p, q) \mid p \in A_1 \textit{ or } q \in A_2\}$, *then* $L(M) = L(M_1) \cup L(M_2)$
- $A = \{(p, q) \mid p \in A_1 \textit{ and } q \in A_2\}$, *then* $L(M) = L(M_1) \cap L(M_2)$
- $A = \{(p, q) \mid p \in A_1 \textit{ and } q \notin A_2\}$, *then* $L(M) = L(M_1) - L(M_2)$

Proof...

[M] Sect 2.2

not substring *aa*

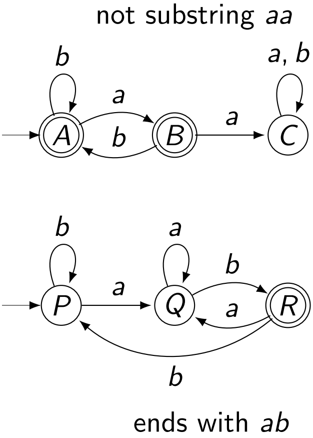


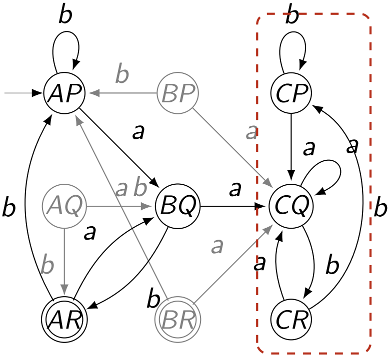




ends with *ab*

[M] E 2.16

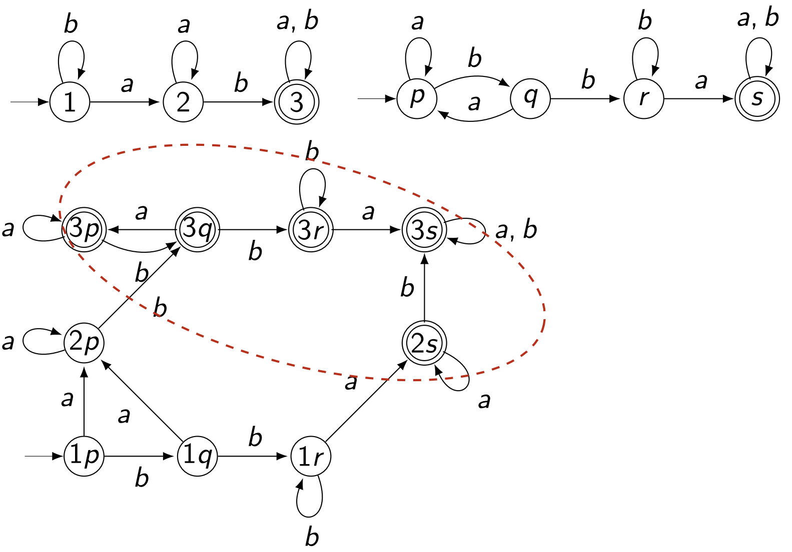
# Example: intersection 'and' (product construction)



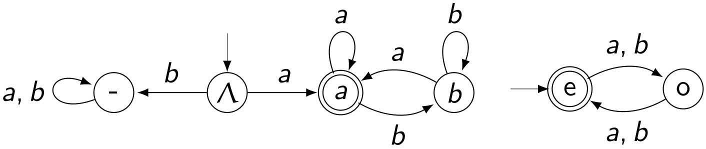

[M] E 2.16

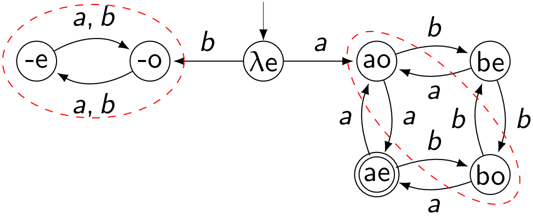
# Example: union, contain either *ab* or *bba*

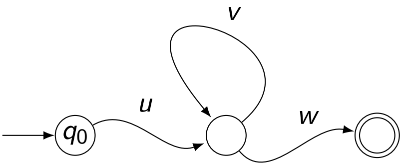



[M] E. 2.18, see also ↪subset construction

$L = \{\ w \in \{a, b\}^* \mid w$ begint en eindigt met een $a$, en $|w|$ is even $\}$






[M] Fig. 2.28

*Regular language* is language accepted by an FA.

**Theorem**

*Suppose $L$ is a language over the alphabet $\Sigma$. If $L$ is accepted by a finite automaton $M$, and if $n$ is the number of states of $M$, then*

$\forall$ *for every $x \in L$*
*satisfying $|x| \geqslant n$*

$\exists$ *there are three strings $u$, $v$, and $w$,*
*such that $x = uvw$ and the following three conditions are true:*
*(1) $|uv| \leqslant n$,*
*(2) $|v| \geqslant 1$*

$\forall$ *and (3) for all $m \geqslant 0$, $uv^m w$ belongs to $L$*

[M] Thm. 2.29

In other words:

**Theorem**

$\forall$ *For every regular language* $L$

$\exists$ *there exists a constant* $n \geqslant 1$

*such that*

$\forall$ *for every* $x \in L$

*with* $|x| \geqslant n$

$\exists$ *there exists a decomposition* $x = uvw$

*with (1)* $|uv| \leqslant n$,

*and (2)* $|v| \geqslant 1$

*such that*

$\forall$ *(3) for all* $m \geqslant 0$, $uv^m w \in L$

if $L = L(M)$ then $n = |Q|$.

[M] Thm. 2.29

In other words:

**Theorem**

*If* $L$ *is a regular language, then*

$\exists$ *there exists a constant* $n \geqslant 1$

*such that*

$\forall$ *for every* $x \in L$

*with* $|x| \geqslant n$

$\exists$ *there exists a decomposition* $x = uvw$

*with (1)* $|uv| \leqslant n$,

*and (2)* $|v| \geqslant 1$

*such that*

$\forall$ *(3) for all* $m \geqslant 0$, $uv^m w \in L$

if $L = L(M)$ then $n = |Q|$.

Introduction to Logic: $p \to q \iff \neg q \to \neg p$

**Theorem**

*If*

$\forall$ *for every* $n \geqslant 1$

$\exists$ *there exists* $x \in L$

*with* $|x| \geqslant n$

*such that*

$\forall$ *for every decomposition* $x = uvw$

*with (1)* $|uv| \leqslant n$,

*and (2)* $|v| \geqslant 1$

$\exists$ *(3) there exists* $m \geqslant 0$,

*such that*

$uv^m w \notin L$

*then L is not a regular language.*

[M] Thm. 2.29

**Example**

$L = \{a^i b^i \mid i \geqslant 0\}$ is not accepted by FA.

[M] E 2.30

Proof: by contradiction

We prove that the language $L = \{a^i b^i \mid i \geqslant 0\}$ is not regular, by contradiction.

Assume that $L = \{a^i b^i \mid i \geqslant 0\}$ is accepted by FA with $n$ states.
Take $x = a^n b^n$. Then $x \in L$, and $|x| = 2n \geqslant n$.
Thus there exists a decomposition $x = uvw$ such that $|uv| \leqslant n$ with $v$ nonempty, and $uv^m w \in L$ for every $m$.
Whatever this decomposition is, $v$ consists of $a$'s only. Consider $m = 0$. Deleting $v$ from the string $x$ will delete a number of $a$'s. So $uv^0 w$ is of the form $a^{n'} b^n$ with $n' < n$.
This string is not in $L$; a contradiction. ($m \geqslant 2$ would also yield contradiction)

So, $L$ is not regular.

**Example**

$L = \{a^i b^i \mid i \geqslant 0\}$ is not accepted by FA.

[M] E 2.30

$AeqB = \{x \in \{a,b\}^* \mid \; n_a(x) = n_b(x) \;\}$

Same argument, or closure properties

FA $M_i = (Q_i, \Sigma, q_i, A_i, \delta_i) \quad i = 1, 2$

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– $A = \{(p, q) \mid p \in A_1$ *and* $q \notin A_2\}$, *then* $L(M) = L(M_1) - L(M_2)$

Proof. . .

[M] Sect 2.2

Exactly the same argument can be used (verbatim) to prove that $L = \text{AeqB}$ is not regular.

We can also apply closure properties of REG to see that AeqB is not regular, as follows.

Assume AeqB is regular. Then also $\text{AnBn} = \text{AeqB} \cap a^* b^*$ is regular, as regular languages are closed under intersection.
This is a contradiction, as we just have argued that AnBn is not regular.
Thus, also AeqB is not regular.

Issues:

- Which $n$? Can I just take $x = aababaabbab$?
- Which $x$? Some $x$ may not yield a contradiction.
- Which decomposition $uvw$? Can I just take $u = a^{10}$, $v = a^{n-10}$, $w = b^n$ ?
- Which $m$? Some $m$ may not yield a contradiction.