Exercise 4.10. Find context-free grammars generating each of the languages below.

a. $\{a^i b^j \mid i \leq j\}$

c. $\{a^i b^j \mid j = 2i\}$

d. $\{a^i b^j \mid i \leq j \leq 2i\}$

e. $\{a^i b^j \mid j \leq 2i\}$

f. $\{a^i b^j \mid j < 2i\}$

Exercise 4.1. In each case below, say what language (a subset of $\{a,b\}^*$) is generated by the context-free grammar with the indicated productions.

b. $S \to SS \mid bS \mid a$

c. $S \to SaS \mid b$

e. $S \to TT \qquad T \to aT \mid Ta \mid b$

f. $S \to aSa \mid bSb \mid aAb \mid bAa \qquad A \to aAa \mid bAb \mid a \mid b \mid \Lambda$

g. $S \to aT \mid bT \mid \Lambda \qquad T \to aS \mid bS$

Exercise 4.3. In each case below, find a CFG generating the given language.

b. The set of even-length strings in $\{a, b\}^*$ with the two middle symbols equal.

c. The set of odd-length strings in $\{a, b\}^*$ whose first, middle, and last symbols are all the same.

Exercise 4.4. In both parts below, the productions in a CFG G are given.
In each part, show first that for every string $x \in L(G)$, $n_a(x) = n_b(x)$; then find a string $x \in \{a, b\}^*$ with $n_a(x) = n_b(x)$ that is not in $L(G)$.

a. $S \to SabS \mid SbaS \mid \Lambda$

b. $S \to aSb \mid bSa \mid abS \mid baS \mid Sab \mid Sba \mid \Lambda$

Exercise 4.9. Suppose that $G_1 = (V_1, \{a,b\}, S_1, P_1)$ and $G_2 = (V_2, \{a,b\}, S_2, P_2)$ are CFGs and that $V_1 \cap V_2 = \varnothing$.

a. It is easy to see that no matter what G_1 and G_2 are, the CFG $G_u = (V_u, \{a,b\}, S_u, P_u)$ defined by $V_u = V_1 \cup V_2$, $S_u = S_1$ and $P_u = P_1 \cup P_2 \cup \{S_1 \to S_2\}$ generates every string in $L(G_1) \cup L(G_2)$. Find grammars G_1 and G_2 (you can use $V_1 = \{S_1\}$ and $V_2 = \{S_2\}$) and a string $x \in L(G_u)$ such that $x \notin L(G_1) \cup L(G_2)$.

b. As in part (a), the CFG $G_c = (V_c, \{a,b\}, S_c, P_c)$ defined by $V_c = V_1 \cup V_2$, $S_c = S_1$ and $P_c = P_1 \cup P_2 \cup \{S_1 \to S_1 S_2\}$ generates every string in $L(G_1)L(G_2)$.
Find grammars G_1 and G_2 (again with $V_1 = \{S_1\}$ and $V_2 = \{S_2\}$) and a string $x \in L(G_c)$ such that $x \notin L(G_1)L(G_2)$.

Exercise 4.9. (continued)

c. The CFG $G^* = (V, \{a,b\}, S, P)$ defined by $V = V_1$, $S = S_1$ and $P = P_1 \cup \{S_1 \to S_1S_1 \mid \Lambda\}$ generates every string in $L(G_1)^*$. Find a grammar G_1 with $V_1 = \{S_1\}$ and a string $x \in L(G^*)$ such that $x \notin L(G_1)^*$.