Section 4

Context-Free Languages

$\langle$assignment$\rangle$ ::= $\langle$variable$\rangle$ = $\langle$expression$\rangle$

$\langle$statement$\rangle$ ::= $\langle$assignment$\rangle$ |
    $\langle$compound-statement$\rangle$ |
    $\langle$if-statement$\rangle$ |
    $\langle$while-statement$\rangle$ | . . .

$\langle$if-statement$\rangle$ ::=
    if $\langle$test$\rangle$ then $\langle$statement$\rangle$ |
    if $\langle$test$\rangle$ then $\langle$statement$\rangle$ else $\langle$statement$\rangle$

$\langle$while-statement$\rangle$ ::=
    while $\langle$test$\rangle$ do $\langle$statement$\rangle$

Definition (well-formed formulas)

. . . by using the construction rules below, and only those, finitely many times:

– every propositional atom $p, q, r, \ldots$ is a wff

– if ϕ is a wff, then so is $(\neg\phi)$

– if ϕ and ψ are wff, then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$,

BNF   Backus Naur form

$\psi ::= p \mid (\neg\psi) \mid (\psi \wedge \psi) \mid (\psi \vee \psi) \mid (\psi \rightarrow \psi)$

M.Huet & M.Ryan, Logic in Computer Science

$AnBn \subseteq \{a, b\}^*$

Example

– $\Lambda \in AnBn$
– for every $x \in AnBn$, also $axb \in AnBn$

$Pal \subseteq \{a, b\}^*$

Example

– $\Lambda, a, b \in Pal$
– for every $x \in Pal$, also $axa, bxb \in Pal$

[M] E 1.18

$Balanced \subseteq \{(,)\}^*$

Example

– $\Lambda \in Balanced$
– for every $x, y \in Balanced$, also $xy \in Balanced$
– for every $x \in Balanced$, also $(x) \in Balanced$

$Expr \subseteq \{a, +, *, (,)\}$

Example

– $a \in Expr$
– for every $x, y \in Expr$, also $x + y \in Expr$ and $x * y \in Expr$
– for every $\in Expr$, als $(x) \in Expr$

[M] E 1.19

Example

– $\Lambda \in AnBn$ (basis)

– for every $x \in AnBn$, also $axb \in AnBn$ (induction)

$S \to \Lambda$
$S \to aSb$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aa\,bb$$
$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaa\,bbb$$

if $S \Rightarrow^* x$ then also $S \Rightarrow^* axb$

Example

– Λ, a, $b \in Pal$
– for every $x \in Pal$, also axa, $bxb \in Pal$

$$S \to \Lambda \mid a \mid b$$
$$S \to aSa$$
$$S \to bSb$$

$$S \Rightarrow aSa \Rightarrow aaSaa \Rightarrow aabSbaa \Rightarrow aababaa$$

$AnBn = \{ a^n b^n \mid n \geqslant 0 \}$

variants

$\{ a^n b^{n+1} \mid n \geqslant 0 \}$

$S \to b$ (end with extra b)
$S \to aSb$

$\{ a^i b^j \mid i \leqslant j \}$

$S \to \Lambda$
$S \to aSb \mid Sb$ (free b's)

$\{ a^i b^j \mid i \neq j \}$

$S \to A \mid B$ (choice!)
$A \to aAb \mid aA \mid a$ $(i > j)$
$B \to aBb \mid Bb \mid b$ $(i < j)$

Example

– $a \in Expr$
– for every $x, y \in Expr$, also $x + y \in Expr$ and $x * y \in Expr$
– for every $\in Expr$, als $(x) \in Expr$

$S \to a \mid S + S \mid S * S \mid (S)$
derivation(s) for $a + (a * a)$ and $a + a * a \ldots$
ambiguity
[M] E 4.2

$NonPal \subseteq \{a, b\}^*$
$x = abbbaaba \in NonPal$
[M] E 4.3

$NonPal \subseteq \{a, b\}^*$
$x = abbbaaba \in NonPal$

Example

– for every $A \in \{a, b\}^*$, aAb and bAa are elements of $NonPal$
– for every S in $NonPal$, aSa and bSb are in $NonPal$

$A \rightarrow \Lambda \mid aA \mid bA$
$S \rightarrow aAb \mid bAa \mid aSa \mid bSb$
[M] E 4.3

$NonPal \subseteq \{a, b\}^*$
$x = abbbaaba \in NonPal$

Example

– for every $A \in \{a, b\}^*$, aAb and bAa are elements of $NonPal$
– for every S in $NonPal$, aSa, bSb, aSb and bSa are in $NonPal$

$A \to \Lambda \mid aA \mid bA$
$S \to aAb \mid bAa \mid aSa \mid bSb \mid aSb \mid bSa$
[M] E 4.3

alphabet $\{ 1, 2, 5, = \}$

$\{ x{=}y \mid x \in \{1, 2\}^*, y \in \{5\}^*, n_1(x) + 2n_2(x) = 5n_5(y) \}$

$n_\sigma(x)$ number of σ occurrences in x

212=5   22222=55 $12(122)^3 2{=}5^4$

The problem with most solutions is that when read from left to right the initial string over $\{0, 1\}$ cannot always be chopped into part with exact value 5, without chopping the symbol 2.
The solution is like a finite automaton, which reads $1, 2$ and 'saves' the values until the value 5 is reached, then we write a 5 to the right.

$\Sigma = \{ 1, 2, 5, = \}$

variables S_i, $0 \leqslant i \leqslant 4$

axiom S_0

productions

$S_0 \to 1S_1 \mid 2S_2$
$S_1 \to 1S_2 \mid 2S_3$
$S_2 \to 1S_3 \mid 2S_4$
$S_3 \to 1S_4 \mid 2S_05$
$S_4 \to 1S_05 \mid 2S_15$
$S_0 \to =$

Definition

context-free grammar (CFG) 4-tuple $G = (V, \Sigma, S, P)$

– V alphabet *variables* / *nonterminals*

– Σ alphabet *terminals* disjoint $V \cap \Sigma = \varnothing$

– $S \in V$ *axiom*, *start symbol*

– P finite set rules, *productions*

of the form $A \to \alpha$, $A \in V$, $\alpha \in (V \cup \Sigma)^*$

derivation step $\alpha = \alpha_1 A \alpha_2 \Rightarrow_G \alpha_1 \gamma \alpha_2 = \beta$ for $A \to \gamma \in P$

Definition

language generated by G

$L(G) = \{ x \in \Sigma^* \mid S \Rightarrow_G^* x \}$

[M] Def 4.6 & 4.7

NonPal, its grammar components

$$A \to \Lambda \mid aA \mid bA$$
$$S \to aAb \mid bAa \mid aSa \mid bSb$$

variables $V = \{ S, A \}$
terminals $\Sigma = \{ a, b \}$
axiom S
productions
$P = \{A \to \Lambda, A \to aA, A \to bA, S \to aAb, S \to bAa, S \to aSa, S \to bSb\}$

$\Rightarrow^*_G$ is the *transitive and reflexive closure* of $\Rightarrow_G$

zero, one or more steps

general case $\alpha = \alpha_0 \Rightarrow \alpha_1 \Rightarrow \ldots \Rightarrow \alpha_n = \beta$

$\alpha \Rightarrow^*_G \beta$ iff there are strings $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that
- $\alpha_0 = \alpha$
- $\alpha_n = \beta$
- $\alpha_i \Rightarrow \alpha_{i+1}$ for $0 \leqslant i < n$.

special case $n = 0$ $\alpha = \alpha_0 = \beta$

Why 'context-free'

Variables can be rewritten regardless of context

Lemma

If $u_1 \Rightarrow^* v_1$ *and* $u_2 \Rightarrow^* v_2$*, then* $u_1 u_2 \Rightarrow^* v_1 v_2$*.*

Lemma

If $u \Rightarrow^* v_1 v v_2$ *and* $v \Rightarrow^* w$*, then* $u \Rightarrow^* v_1 w v_2$*.*

Lemma

If $u \Rightarrow^* v$ *and* $u = u_1 u_2$*,*
then $v = v_1 v_2$ *such that* $u_1 \Rightarrow^* v_1$ *and* $u_2 \Rightarrow^* v_2$*.*

$AeqB = \{ x \in \{a, b\}^* \mid n_a(x) = n_b(x) \}$

aaabbb, *ababab*, *aababb*, . . .

[M] E 4.8

From lecture 6:

– Even number of both *a* and *b*

two letters together

aa and *bb* keep both numbers even [odd]

ab and *ba* switch between even and odd, for both numbers

$(aa + bb + (ab + ba)(aa + bb)^*(ab + ba))^*$

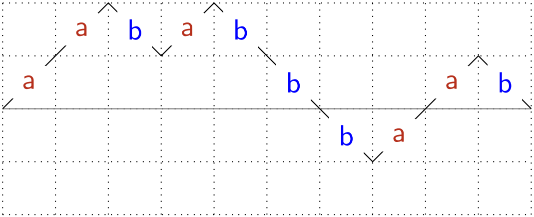
[M] E 3.4

$AeqB = \{ \; x \in \{a, b\}^* \mid n_a(x) = n_b(x) \; \}$

aaabbb, *ababab*, *aababb*, . . .

[M] E 4.8

$$AeqB = \{\ x \in \{a, b\}^* \mid n_a(x) = n_b(x)\ \}$$



$AeqB = \{ x \in \{a, b\}^* \mid n_a(x) = n_b(x) \}$

$aaabbb, ababab, aababb, \ldots$

$$\begin{aligned} S &\to \Lambda \mid aB \mid bA \\ A &\to aS \mid bAA \\ B &\to bS \mid aBB \end{aligned}$$

A generates $n_a(x) = n_b(x) + 1$

B generates $n_a(x) + 1 = n_b(x)$

$S \Rightarrow aB \Rightarrow aaBB \Rightarrow aabSB \Rightarrow \ldots$ (different options)

(1) $aabB \Rightarrow aabaBB \Rightarrow aababSB \Rightarrow aababB \Rightarrow aababbS \Rightarrow aababb$

(2) $\ldots$ (ambiguous, later)

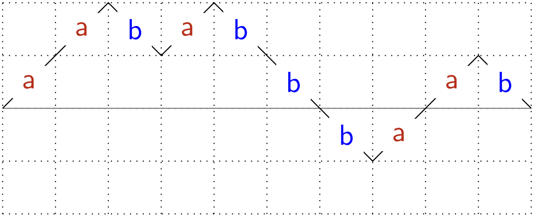
[M] E 4.8

ABOVE

When a string has multiple variables, like $aabSB$ in the above example, then we are not forced to rewrite the first variable, we can as well rewrite another one.

Thus we can do $aab\underline{S}B \Rightarrow aabB$, but also $aabS\underline{B} \Rightarrow aabSaBB$, for instance.

$AeqB = \{\, x \in \{a, b\}^* \mid n_a(x) = n_b(x) \,\}$



$S \to \Lambda \mid aSb \mid bSa \mid SS$

$S \Rightarrow SS \Rightarrow a_1 S b_6 S \Rightarrow a_1 a_2 S b_3 S b_6 S \Rightarrow \ldots$

$S \Rightarrow a_1 S b_{10} \Rightarrow \ldots$

[M] Exercise 1.66

$L_1 = \{ a^i b^j c^k \mid i = j + k \}$ $\quad$ $aaa\,b\,cc$

$L_1 = \{ a^i b^j c^k \mid i = j + k \}$ $\quad aaa\,b\,cc$

generate as $\quad a^{k+j}\ b^j\ c^k = \underbrace{a^k\ \underbrace{a^j\ b^j}\ c^k}$

$$S \to aSc \mid T$$
$$T \to aTb \mid \Lambda$$

$$S \Rightarrow aSc \Rightarrow aaScc \Rightarrow aaTcc \Rightarrow aaaTbcc \Rightarrow aaabcc$$

$L_1 = \{ a^i b^j c^k \mid i = j + k \}$ $\quad aaa\,b\,cc$

generate as $\quad a^{k+j}\ b^j\ c^k = \underbrace{a^k\ \underbrace{a^j\ b^j}\ c^k}$

$S \to aSc \mid T$
$T \to aTb \mid \Lambda$

$S \Rightarrow aSc \Rightarrow aaScc \Rightarrow aaTcc \Rightarrow aaaTbcc \Rightarrow aaabcc$

$L_2 = \{ a^i b^j c^k \mid j = i + k \}$ $\quad a\,bbb\,cc$

$L_1 = \{ a^i b^j c^k \mid i = j + k \}$ $\quad aaa\, b\, cc$

generate as $\quad a^{k+j}\ b^j\ c^k = \underbrace{a^k\ \underbrace{a^j\ b^j}\ c^k}$

$S \to aSc \mid T$
$T \to aTb \mid \Lambda$

$S \Rightarrow aSc \Rightarrow aaScc \Rightarrow aaTcc \Rightarrow aaaTbcc \Rightarrow aaabcc$

$L_2 = \{ a^i b^j c^k \mid j = i + k \}$ $\quad a\, bbb\, cc$

generate as $\quad a^i\ b^{i+k}\ c^k = \underbrace{a^i\ b^i}\ \underbrace{b^k\ c^k}$

$S \to XY$ (concatenate)
$X \to aXb \mid \Lambda$
$Y \to bYc \mid \Lambda$

$S \Rightarrow \underline{X}\ Y \Rightarrow a\underline{X}b\ Y \Rightarrow ab\ \underline{Y} \Rightarrow ab\ b\underline{Y}c \Rightarrow ab\ bb\underline{Y}cc \Rightarrow abbbcc$
$S \Rightarrow X\ \underline{Y} \Rightarrow \underline{X}\ bYc \Rightarrow aXb\ b\underline{Y}c \Rightarrow a\underline{X}b\ bbYcc \Rightarrow ab\ bb\underline{Y}cc \Rightarrow abbbcc$

(a priori there is no prescribed order rewriting X or Y)

Using building blocks

Theorem

If L_1, L_2 are CFL, then so are $L_1 \cup L_2$, $L_1 L_2$ and L_1^*.

[M] Thm 4.9

Using building blocks

Theorem

If L_1, L_2 are CFL, then so are $L_1 \cup L_2$, $L_1 L_2$ and L_1^*.

$G_i = (V_i, \Sigma, S_i, P_i)$, having no variables in common.
[M] Thm 4.9

Using building blocks

Theorem

If L_1, L_2 are CFL, then so are $L_1 \cup L_2$, $L_1 L_2$ and L_1^*.

$G_i = (V_i, \Sigma, S_i, P_i)$, having no variables in common.

Construction

$G = (V_1 \cup V_2 \cup \{S\}, \Sigma, S, P)$, new axiom S
– $P = P_1 \cup P_2 \cup \{S \to S_1, S \to S_2\}$ $L(G) = L(G_1) \cup L(G_2)$
– $P = P_1 \cup P_2 \cup \{S \to S_1 S_2\}$ $L(G) = L(G_1) L(G_2)$

$G = (V_1 \cup \{S\}, \Sigma, S, P)$, new axiom S
– $P = P_1 \cup \{S \to SS_1, S \to \Lambda\}$ $L(G) = L(G_1)^*$

[M] Thm 4.9

$$L_0 = \{ a^i b^j c^k \mid j = i + k \} = \{ a^i b^{i+k} c^k \mid j = i + k \}$$
$$= \{ \underbrace{a^i b^i} \underbrace{b^k c^k} \mid j = i + k \}$$

$$S_0 \to XY \quad X \to aXb \mid \Lambda \quad Y \to bYc \mid \Lambda$$

$$L = \{ a^i b^j c^k \mid j \neq i + k \}$$

[M] E 4.10

$$L_0 = \{\, a^i b^j c^k \mid j = i+k \,\} = \{\, a^i b^{i+k} c^k \mid j = i+k \,\}$$
$$= \{\, \underbrace{a^i b^i}\underbrace{b^k c^k} \mid j = i+k \,\}$$

$$S_0 \to XY \quad X \to aXb \mid \Lambda \quad Y \to bYc \mid \Lambda$$

$$L = \{\, a^i b^j c^k \mid j \neq i+k \,\} = L_1 \cup L_2$$
$$S \to S_1 \mid S_2$$

$$L_1 = \{\, a^i b^j c^k \mid j > i+k \,\}$$
$$L_2 = \{\, a^i b^j c^k \mid j < i+k \,\}$$

[M] E 4.10

$$L_0 = \{\, a^i b^j c^k \mid j = i+k \,\} = \{\, a^i b^{i+k} c^k \mid j = i+k \,\}$$
$$= \{\, \underbrace{a^i b^i}\, \underbrace{b^k c^k} \mid j = i+k \,\}$$

$$S_0 \to XY \qquad X \to aXb \mid \Lambda \qquad Y \to bYc \mid \Lambda$$

$L = \{\, a^i b^j c^k \mid j \neq i+k \,\} = L_1 \cup L_2$

$S \to S_1 \mid S_2$

$L_1 = \{\, a^i b^j c^k \mid j > i+k \,\}$

$S_1 \to X_1 b Y_1$

$X_1 \to aX_1 b \mid X_1 b \mid \Lambda$

$Y_1 \to bY_1 c \mid bY_1 \mid \Lambda$

$L_2 = \{\, a^i b^j c^k \mid j < i+k \,\}$

[M] E 4.10

$$L_0 = \{ a^i b^j c^k \mid j = i + k \} = \{ a^i b^{i+k} c^k \mid j = i + k \}$$
$$= \{ \underbrace{a^i b^i}\, \underbrace{b^k c^k} \mid j = i + k \}$$

$$S_0 \to XY \quad X \to aXb \mid \Lambda \quad Y \to bYc \mid \Lambda$$

$L = \{ a^i b^j c^k \mid j \neq i + k \} = L_1 \cup L_2$

$S \to S_1 \mid S_2$

$L_1 = \{ a^i b^j c^k \mid j > i + k \}$

$S_1 \to X_1 b Y_1$
$X_1 \to aX_1 b \mid X_1 b \mid \Lambda$
$Y_1 \to bY_1 c \mid bY_1 \mid \Lambda$

$L_2 = \{ a^i b^j c^k \mid j < i + k \}$

$S_2 \to aX_2 Y_2 \mid X_2 Y_2 c$
$X_2 \to aX_2 b \mid aX_2 \mid \Lambda$
$Y_2 \to bY_2 c \mid Y_2 c \mid \Lambda$

[M] E 4.10

ABOVE

De uitwerking uit het boek is wat te ingewikkeld, dat hebben we hier wat ingekort.

$S \to S_1 \mid S_2$	union
$S \to S_1 S_2$	concatenation
$S \to SS_1 \mid \Lambda$	star

CFG for $\varnothing$...
CFG for $\{\sigma\}$...

Example

$L = bba(ab)^* + (ab + ba^*b)^*ba$

[M] E 4.11