EXAM AUTOMATA THEORY
Thursday 21 December 2023, 09:00 - 12:00

This exam consists of eight exercises, where [x pt] indicates how many points can be earned per exercise. A total of 100 points can be earned.
It is important to provide an explanation or motivation when a question asks for it.
A finite automaton in this exam (without further addition), refers to a deterministic finite automaton without Λ-transitions (which is elsewhere called *DFA*).

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1. [8 pt] Let

$$L = \{x \in \{a,b\}^* \quad | \quad n_a(x) + 3 \cdot n_b(x) \equiv 0 \pmod 4\ \}$$

For example, $aaaa \in L$, but $bb \notin L$.

Draw a finite automaton M, such that $L(M) = L$ with at most five states. If your automaton has more than five states, use the minimization algorithm – *on scratch paper*, to reduce the number of states.

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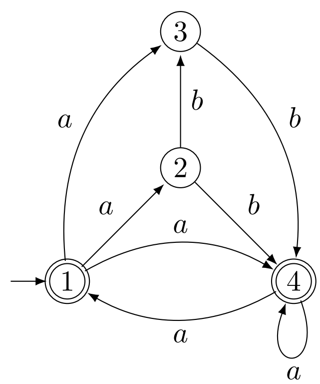
2. [9 pt] Use the subset construction, i.e., Theorem 3.18 in the book, to transform the non-deterministic finite automaton M below into an equivalent finite automaton.



Remove unreachable states (if any) and draw only the resulting automaton. Make sure that the names of the states of M can still be recognized in your answer.

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3. [20 pt] The pumping lemma for regular languages reads as follows:

   Suppose L is a language over the alphabet Σ.
   If L is accepted by a finite automaton M,
   and if n is the number of states of M, then:
   $\forall$ for every $x \in L$
   satisfying $|x| \geq n$
   $\exists$ there are three strings u, v, and w,
   such that $x = uvw$ and the following conditions are true:
   (1) $|uv| \leq n$,
   (2) $|v| \geq 1$
   $\forall$ and (3) for all $m \geq 0$, $uv^m w$ belongs to L.

   Now let
   $$L_1 = \{ \ a^i b^j \mid i \neq j \}$$

   For example, $a^5 b^4 \in L_1$. Let us assume that $n \geq 2$ for this exercise.

   (a) For each of the following four strings x_1, x_2, x_3, x_4, indicate whether it is suitable for establishing a contradiction with the pumping lemma. Furthermore, for each of the strings x_i that is **not** suitable, indicate why not, for example, via a concrete decomposition uvw of x_i that does satisfy the pumping lemma.
   If x_i is suitable for contradicting the pumping lemma, then you don't have to explain that.
   $$x_1 = a^{n+1} b^n$$
   $$x_2 = a^{n!} b^{(n+1)!}$$
   $$x_3 = aaaaabbbb$$
   $$x_4 = b^{2n}$$

   (b) Give or describe the elements of $L_1 / a^2 b^2$.

   (c) Give or describe the elements of (the equivalence class) $[a^2 b^2]$, i.e., all elements of $\{a, b\}^*$ that are *indistinguishable* from $a^2 b^2$ with respect to L_1.

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4. [9 pt]

   Consider the two regular expressions:
   $$r = (bb + (a + ba)a^* b)^* (b + ba)a^*$$
   $$s = (bb + b)^* (b + (a + ba)a^*)$$

   (a) Find a minimum-length string corresponding to r but not to s.

   (b) Find a minimum-length string corresponding to s but not to r.

   (c) Find a minimum-length string corresponding to both r and s.

   (d) Find a minimum-length string in $\{a, b\}^*$ corresponding to neither r nor s.

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5. [14 pt] Let again

$$L_1 = \{ a^i b^j \mid i \neq j \}$$

   (a) Give a context-free grammar G_1, such that $L(G_1) = L_1$. Try to ensure that G_1 is unambiguous. If you do not succeed in this, then you can still earn most of the points.

   If your context-free grammar is ambiguous, then give two different derivation trees for a string $x \in L_1$.

   (b) Give a derivation in your context-free grammar G_1 for the string $x = abbb$.

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6. [11 pt] A context-free grammar $G = (V, \Sigma, S, P)$ is said to be in *Chomsky normal form*, if each production in G is of one the following two forms:

$$\begin{array}{lll} A & \to & BC \qquad \text{with } A, B, C \in V \\ A & \to & \sigma \qquad \text{with } A \in V \text{ and } \sigma \in \Sigma \end{array}$$

Suppose that G is indeed in Chomsky normal form, that $\Sigma = \{a, b\}$, and that $V = \{S\}$, i.e., G has only one variable.

What could $L(G)$ be in this case, depending on the productions in P? Consider all possible cases.

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7. [18 pt] Let

$$L = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i + k < j\}$$

This exercise deals with the complement L' of L, i.e., the language of strings over $\{a, b, c\}$ that are not in L.

(a) Give the first four elements in the canonical (shortlex) order of L'.

(b) Draw a pushdown automaton M, such that $L(M) = L'$.

This pushdown automaton must be based directly on the properties of the language. It should, therefore, not be the result of a standard construction for, for example, converting a context-free grammar into a pushdown automaton.

Try to ensure that M is deterministic and does not contain any Λ-transitions. If you do not succeed in this, you can still earn most of the points.

Also explain how M uses its states and stack to accept precisely the right language.

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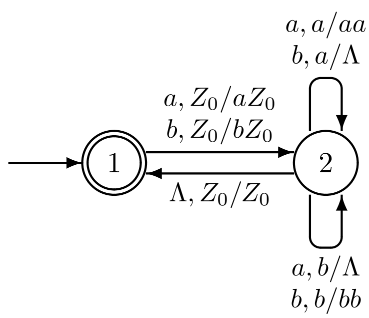
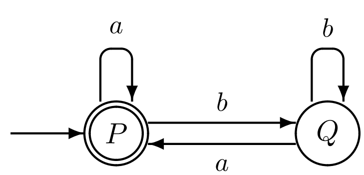
8. [11 pt] Consider the following pushdown automaton M_1:



(a) Formally, a pushdown automaton is a 7-tuple $(Q, \Sigma, \Gamma, q_0, Z_0, A, \delta)$. What are, for the concrete pushdown automaton M_1: Q, Σ, Γ, q_0 and A (as far as you can deduce from the picture)?

(b) Now also consider the following finite automaton M_2:



In the lectures, we have discussed a construction to combine a pushdown automaton M_1 and a finite automaton M_2, such that the resulting pushdown automaton M accepts $L(M_1) \cap L(M_2)$. Apply this construction to the two automata M_1 and M_2 above.

`end of exam`