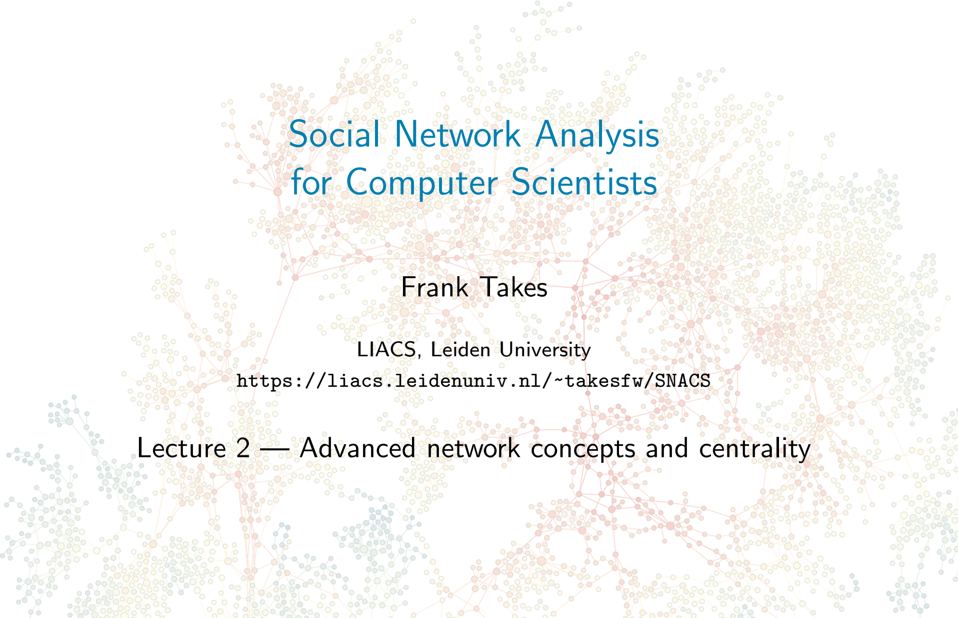


Social Network Analysis for Computer Scientists

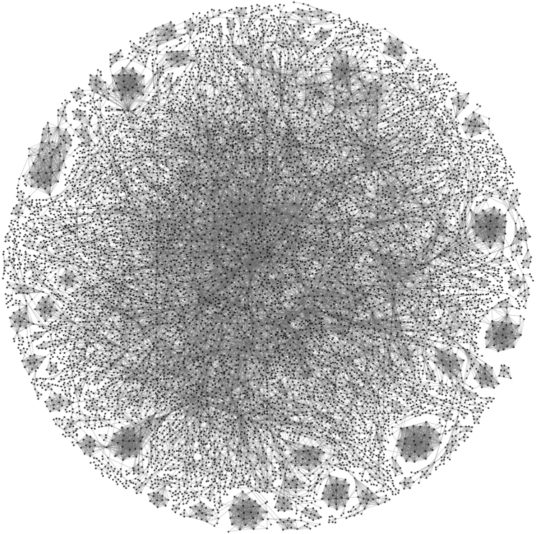
Frank Takes

LIACS, Leiden University
`https://liacs.leidenuniv.nl/~takesfw/SNACS`

Lecture 2 — Advanced network concepts and centrality

Recap

Networks



Notation

Concept	**Symbol**
Network (graph)	$G = (V, E)$
Nodes (objects, vertices, . . .)	V
Links (ties, relationships, . . .)	E
Directed — $E \subseteq V \times V$ — "links"	
Undirected — "edges"	
Number of nodes — $\lvert V \rvert$	n
Number of edges — $\lvert E \rvert$	m
Degree of node u	$deg(u)$
Distance from node u to v	$d(u, v)$

Real-world networks

1 Sparse networks	density
2 Fat-tailed power-law degree distribution	degree
3 Giant component	components
4 Low pairwise node-to-node distances	distance
5 Many triangles	clustering coefficient

- Many examples: communication networks, citation networks, collaboration networks (Erdös, Kevin Bacon), protein interaction networks, information networks (Wikipedia), webgraphs, financial networks (Bitcoin) . . .

Advanced concepts

Advanced concepts

- Assortativity
- Reciprocity
- Power law exponent
- Planar graphs
- Complete graphs
- Subgraphs
- Trees
- Spanning trees
- Diameter
- Bridges
- Graph traversal

Assortativity

- **Assortativity**: extent to which “similar” nodes attract each other
  Value close to -1 if dissimilar nodes more often attract each other
  Value close to 1 if similar nodes more often attract each other

Assortativity

- **Assortativity**: extent to which "similar" nodes attract each other
  Value close to -1 if dissimilar nodes more often attract each other
  Value close to 1 if similar nodes more often attract each other
- Degree assortativity: nodes with a similar degree connect more frequently
- Attribute assortativity: nodes with similar attributes attract each other
- Influence on connectivity of network, spreading of information, etc.
- Social networks: **homophily**
- Complex networks: **mixing patterns**

Degree assortativity

Figure: Degree **assortativity** (left) and degree **disassortativity** (right)

Image: Estrada et al., Clumpiness mixing in complex networks, J. Stat. Mech. Theor. Exp. P03008 (2008).

Attribute assortativity

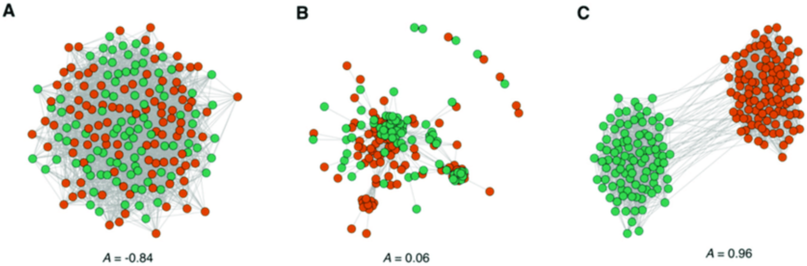


Figure: Attribute assortativity

Image: Moya-García, A. et al. Identification of New Toxicity Mechanisms ... *Genes*, 13(7), 1292, 2022.

Reciprocity

- **Reciprocity**: measure of the likelihood of nodes in a directed network to be mutually linked
- Let $m_{<->}$ be the number of links in the directed network for which there also exists a symmetric counterpart:

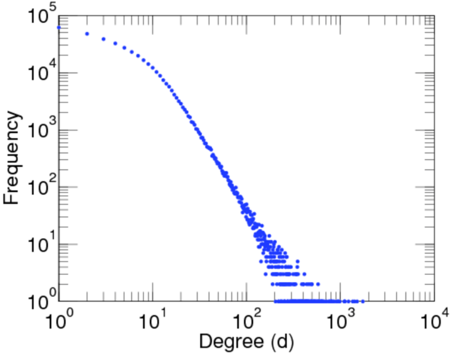
$$m_{<->} = |\{(u, v) \in E : (v, u) \in E\}|$$

- Reciprocity r is then the fraction of links that is symmetric:

$$r = \frac{m_{<->}}{m}$$

- Measures the extent to which relationships are mutual
- Useful to compare between networks

Power law degree distribution



Source: `http://konect.cc/networks/citeseer/`

Power law exponent in undirected networks

- The probability p_k of a node having degree k depends on the power law exponent γ:

$$p_k \sim k^{-\gamma}$$

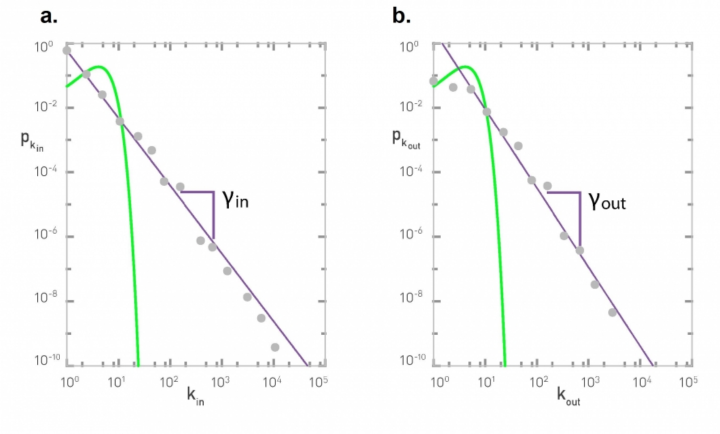
- This means that

$$\log p_k \sim -\gamma \log k$$

  And as such, the straight line in log-log scale plots is observed.

- In real-world networks, γ has a value of around 2 to 3
- Useful to compare between similar networks

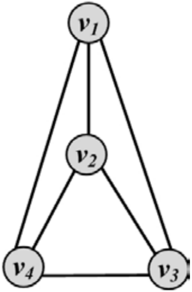
Power law exponent in directed networks



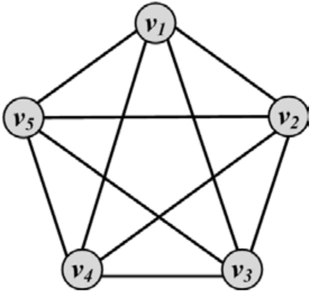
Source: A. Barabasi, Network Science, 2016.

Planar graphs

- **Planar graphs** can be visualized such that no two edges cross each other



(a) Planar Graph



(b) Non-planar Graph

Image: Zafarani et al., Social Media Mining, 2014.

Complete graphs

- In **complete graphs**, all pairs of nodes are connected
- The number of edges m is equal to $\frac{1}{2} \cdot n \cdot (n-1)$

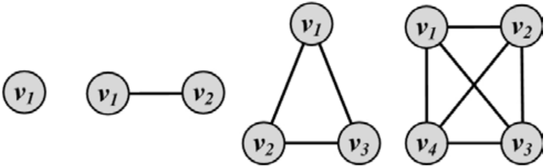


Figure: Complete graphs of size 1, 2, 3 and 4

Image: Zafarani et al., Social Media Mining, 2014.

Ego network

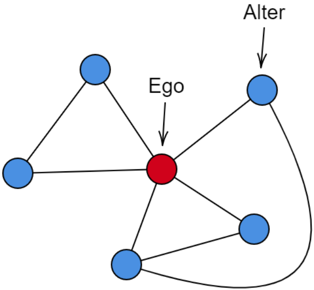


Figure: The **ego network** of a given node in a network consists of the set of nodes containing that node ("Ego") and its direct neighbors ("Alters"), and all edges present between the nodes in this set

Image: Wikipedia "Egocentric network.png", accessed 2022.

Trees

- A **tree** is a graph without cycles
- A set of disconnected trees is called a **forest**
- A tree with n nodes has $m = n - 1$ edges

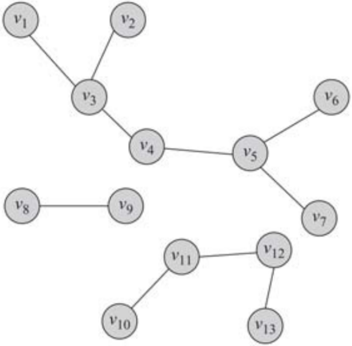


Image: Zafarani et al., Social Media Mining, 2014.

Trees

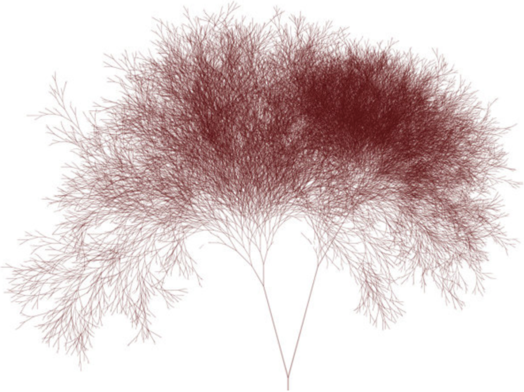


Image: M. Lima, Book of trees: Visualizing branches of knowledge, 2014.

Subgraphs

- Given a graph $G = (V, E)$
- **Subgraph** $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq (E \cap (V' \times V'))$ (subset of the nodes and edges of the original network, commonly used when defining communities or clusters)
- **Subgraph** $G' = (V, E')$ with $E' \subseteq E$ (only edges are left out, commonly used when modelling network evolution)
- Special subgraphs: spanning trees

Spanning trees

- A **spanning tree** is a tree and subgraph of a graph that covers all nodes of the graph
- In weighted graphs, a **minimal** spanning tree is one of minimal edge weight

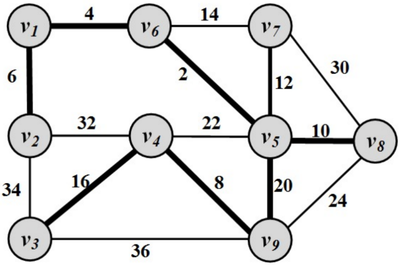


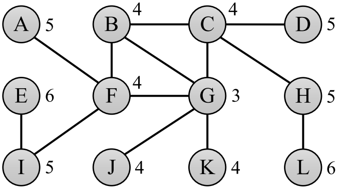
Image: Zafarani et al., Social Media Mining, 2014.

Diameter

- Distance $d(u, v)$ = length of shortest path from u to v
- Diameter $D(G) = \max_{u,v \in V} d(u, v)$ = maximal distance

Diameter

- Distance $d(u,v)$ = length of shortest path from u to v
- Diameter $D(G) = \max_{u,v \in V} d(u,v)$ = maximal distance
- Eccentricity $e(u) = \max_{v \in V} d(u,v)$ = length of longest shortest path from u
- Diameter $D(G) = \max_{u \in V} e(u)$ = maximal eccentricity
- Radius $R(G) = \min_{u \in V} e(u)$ = minimal eccentricity



Bridges

- **Bridge**: an edge whose removal will result in an increase in the number of connected components
- Also called **cut edges**, with applications in community detection

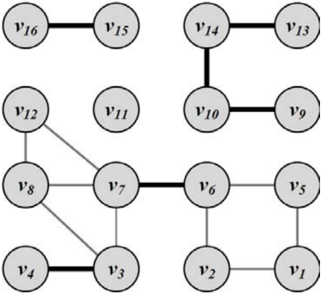


Image: Zafarani et al., Social Media Mining, 2014.

Graph traversal

- Given a network, how can we explore it?
- Specifically: exploration starting from a particular source
- Node **adjacency**: two nodes are adjacent if there is an edge connecting them
- **Neighborhood**: set of nodes adjacent to a node $v \in V$:

$$N(v) = \{w \in V : (v, w) \in E\}$$

- Techniques to iteratively explore neighborhoods: DFS and BFS

Graph traversal: DFS

- Depth First Search (DFS)

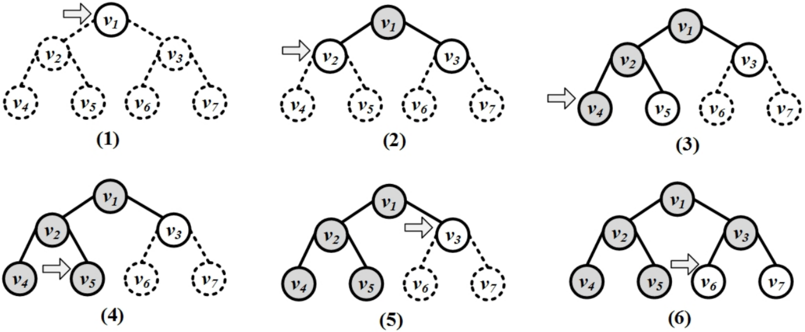
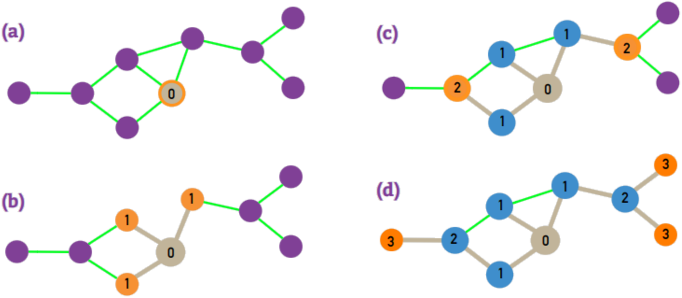


Image: Zafarani et al., Social Media Mining, 2014.

Graph traversal: BFS



Source: A. Barabasi, Network Science, 2016.

Graph traversal: BFS

- **Breadth First Search** (BFS)
- Graph traversal in level-order

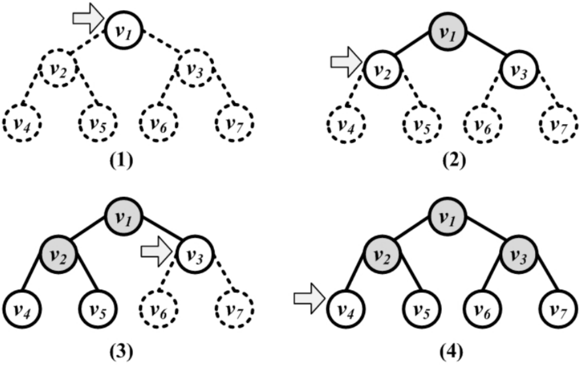


Image: Zafarani et al., Social Media Mining, 2014.

Graph traversal: BFS

- **Breadth First Search** (BFS)
- From source node, create a rooted spanning tree of the graph
- Graph traversal in level-order
- Often implemented using a queue
- BFS considers traversing each of the m edges once, so $O(m)$
- Important for computing various centrality measures

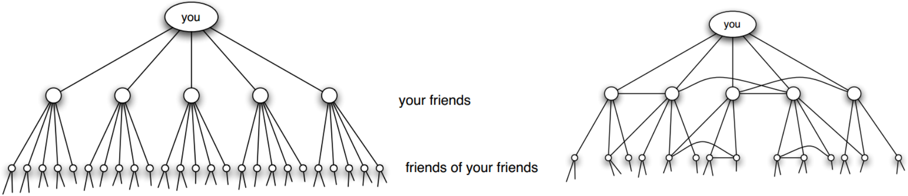


Image: Zafarani et al., Social Media Mining, 2014.

Centrality

Centrality

- Given a social network, which person is most important?
- What is the most important page on the web?
- Which protein is most vital in a biological network?
- Who is the most respected author in a scientific citation network?
- What is the most crucial router in an internet topology network?

Centrality

- **Node centrality**: the importance of a node with respect to the other nodes based on the structure of the network
- **Centrality measure**: computes the centrality value of all nodes in the graph
- For all $v \in V$ a measure M returns a value $C_M(v) \in [0;1]$
- $C_M(v) > C_M(w)$ means that node v is more important than w


GAME OF THRONES
Watermark

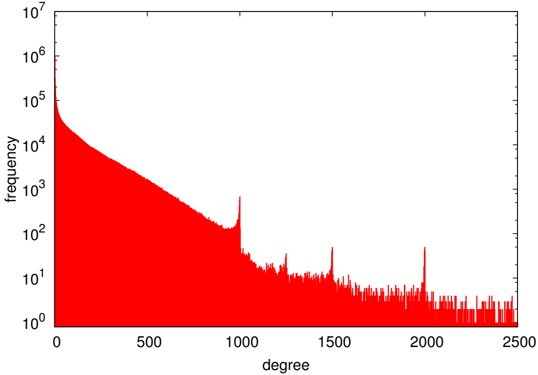
Degree centrality

- Undirected graphs – **degree centrality**: measure the number of adjacent nodes

$$C_d(v) = \frac{deg(v)}{n-1}$$

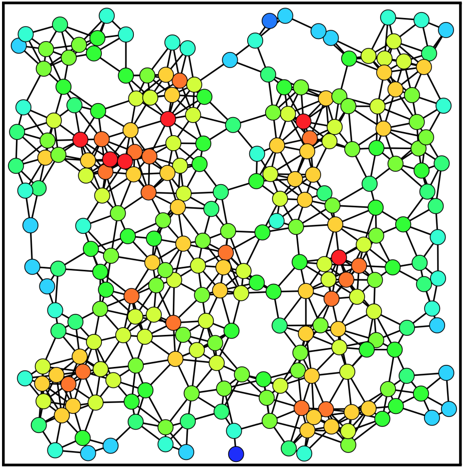
- Directed graphs — indegree centrality and outdegree centrality
- Local measure
- $O(1)$ time to compute

Degree distribution

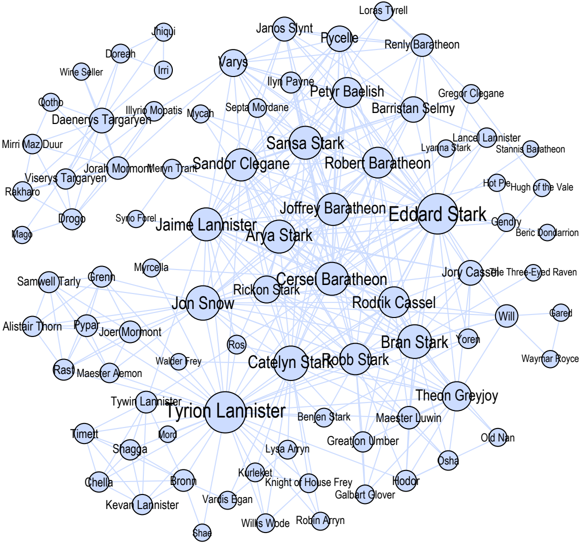


- Not so many distinct values in the lower ranges

Degree centrality



Degree centrality



Closeness centrality

- **Closeness centrality**: based on the average distance to all other nodes
$$C_c(v) = \left(\frac{1}{n-1} \sum_{w \in V} d(v,w) \right)^{-1}$$
- Global distance-based measure
- $O(mn)$ to compute: one BFS in $O(m)$ for each of the n nodes

Closeness centrality

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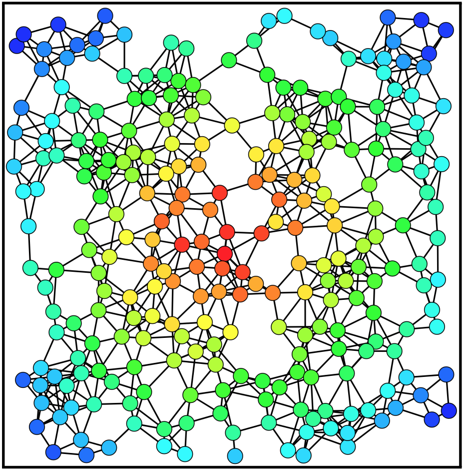
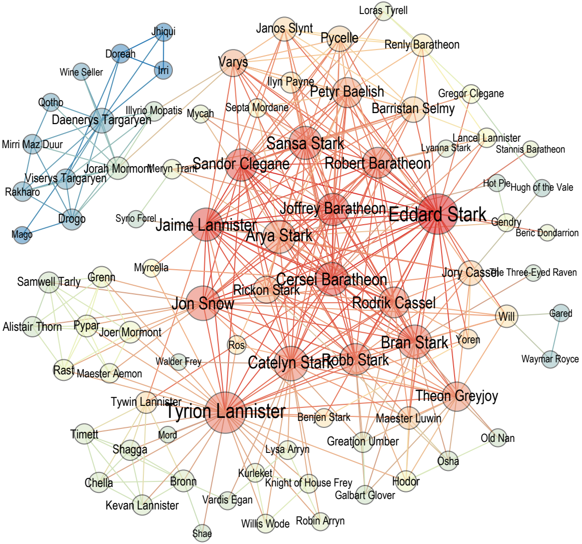
- Global distance-based measure
- $O(mn)$ to compute: one BFS in $O(m)$ for each of the n nodes
- Connected component(s)...
- **Harmonic centrality**: variant of closeness (not normalized)

$$C_h(v) = \sum_{w \in V} \frac{1}{d(w,v)}$$

Closeness centrality



Degree vs. closeness centrality



Betweenness centrality

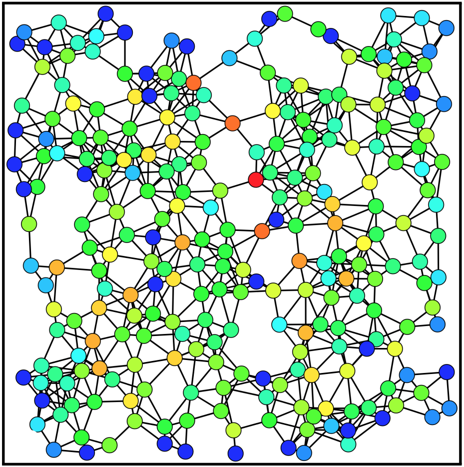
- **Betweenness centrality**: measure the number of shortest paths that run through a node

$$C_b(u) = \sum_{\substack{v,w \in V \\ v \neq w, u \neq v, u \neq w}} \frac{\sigma_u(v,w)}{\sigma(v,w)}$$

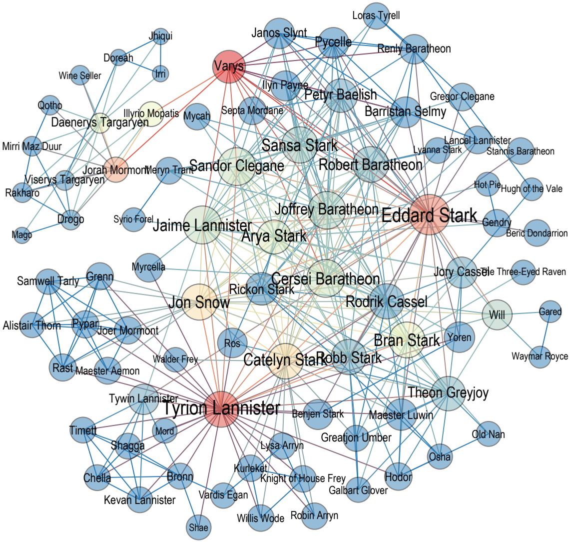
- $\sigma(v,w)$ is the number of shortest paths from v to w
- $\sigma_u(v,w)$ is the number of such shortest paths that run through u
- Divide by largest value to normalize to $[0;1]$
- Global path-based measure
- $O(2mn)$ time to compute (two "BFSes" for each node)

U. Brandes, "A faster algorithm for betweenness centrality", Journal of Mathematical Sociology 25(2): 163–177, 2001

Betweenness centrality



Degree vs. betweeness centrality



Centrality measures compared

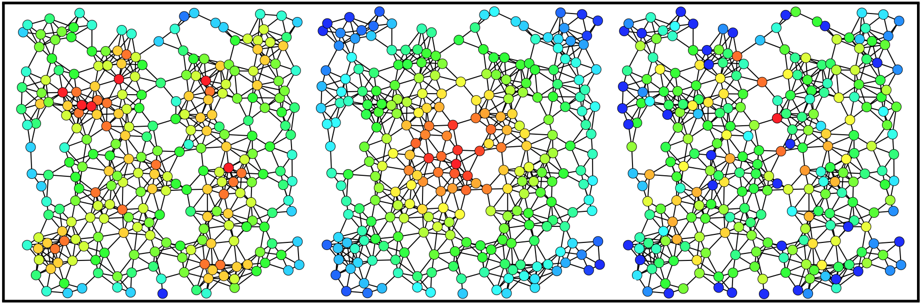


Figure: Degree, closeness and betweenness centrality

Source: "Centrality"' by Claudio Rocchini, Wikipedia `File:Centrality.svg`

Eccentricity centrality

- Node **eccentricity**: length of a longest shortest path (distance to a node furthest away)

$$e(v) = \max_{w \in V} d(v, w)$$

- **Eccentricity centrality**:

$$C_e(v) = \frac{1}{e(v)}$$

- Worst-case variant of closeness centrality
- $O(mn)$ to compute: one BFS in $O(m)$ for each of the n nodes

Eccentricity centrality

- Node **eccentricity**: length of a longest shortest path (distance to a node furthest away)

$$e(v) = \max_{w \in V} d(v, w)$$

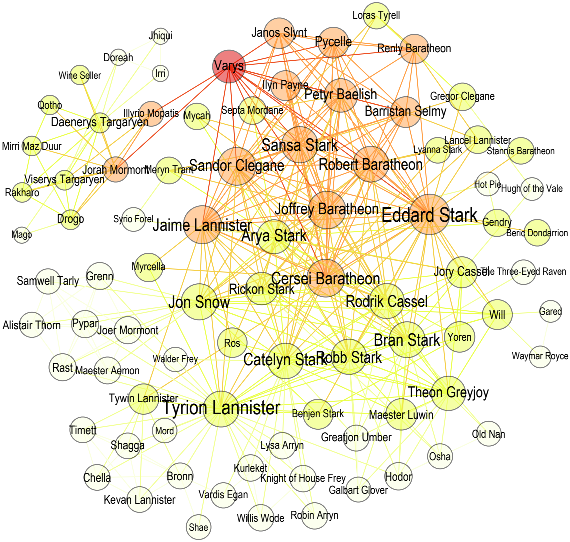
- **Eccentricity centrality**:

$$C_e(v) = \frac{1}{e(v)}$$

- Worst-case variant of closeness centrality
- $O(mn)$ to compute: one BFS in $O(m)$ for each of the n nodes
    - Large optimizations possible using lower and upper bounds, see F.W. Takes and W.A. Kosters, Computing the Eccentricity Distribution of Large Graphs, *Algorithms*, vol. 6, nr. 1, pp. 100-118, 2013.

Eccentricity centrality

Degree vs. eccentricity centrality

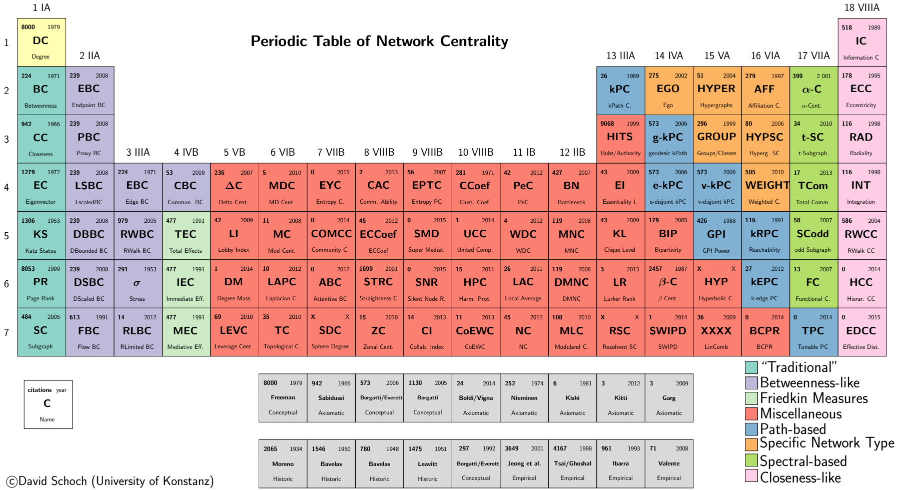


Centrality measures

- Distance/path-based measures:
    - Degree centrality $O(n)$
    - Closeness centrality $O(mn)$
    - Betweenness centrality $O(mn)$
    - Eccentricity centrality $O(mn)$

  (complexity is for computing centralities of all n nodes)
- Many more: Eigenvector centrality, Katz centrality, ...
- Approximating these measures is also possible
- Also: propagation-based centrality measures like PageRank

Periodic table of centrality



Periodic Table of Network Centrality

	1 IA	2 IIA	3 IIIA	4 IVB	5 VB	6 VIB	7 VIIB	8 VIIIB	9 VIIIB	10 VIIIB	11 IB	12 IIB	13 IIIA	14 IVA	15 VA	16 VIA	17 VIIA	18 VIIIA
1	8000 1979 **DC** Degree																	518 1989 **IC** Information C
2	224 1971 **BC** Betweenness	239 2008 **EBC** Endpoint BC											26 1989 **kPC** kPath C.	275 2002 **EGO** Ego	51 2004 **HYPER** Hypergraphs	279 1997 **AFF** Affiliation C.	399 2 001 **α-C** α-Cent.	178 1995 **ECC** Eccentricity
3	942 1966 **CC** Closeness	239 2008 **PBC** Proxy BC											9068 1999 **HITS** Hubs/Authority	573 2006 **g-kPC** geodesic kPath	296 1999 **GROUP** Groups/Classes	80 2006 **HYPSC** Hyperg. SC	34 2010 **t-SC** t-Subgraph	116 1998 **RAD** Radiality
4	1279 1972 **EC** Eigenvector	239 2008 **LSBC** LscaledBC	224 1971 **EBC** Edge BC	53 2009 **CBC** Commun. BC	236 2007 **ΔC** Delta Cent.	5 2010 **MDC** MD Cent.	0 2015 **EYC** Entropy C.	2 2013 **CAC** Comm. Ability	56 2007 **EPTC** Entropy PC	281 1971 **CCoef** Clust. Coef.	42 2012 **PeC** PeC	427 2007 **BN** Bottleneck	43 2009 **EI** Essentiality I.	573 2006 **e-kPC** e-disjoint kPC	573 2006 **v-kPC** v-disjoint kPC	505 2010 **WEIGHT** Weighted C.	17 2013 **TCom** Total Comm.	116 1998 **INT** Integration
5	1306 1953 **KS** Katz Status	239 2008 **DBBC** DBounded BC	979 2005 **RWBC** RWalk BC	477 1991 **TEC** Total Effects	42 2009 **LI** Lobby Index	11 2008 **MC** Mod Cent.	0 2014 **COMCC** Community C.	45 2012 **ECCoef** ECCoef	0 2015 **SMD** Super Mediat.	1 2014 **UCC** United Comp.	4 2012 **WDC** WDC	119 2008 **MNC** MNC	43 2009 **KL** Clique Level	179 2005 **BIP** Bipartivity	426 1988 **GPI** GPI Power	116 1991 **kRPC** Reachability	58 2007 **SCodd** odd Subgraph	586 2004 **RWCC** RWalk CC
6	8053 1999 **PR** Page Rank	239 2008 **DSBC** DScaled BC	291 1953 **σ** Stress	477 1991 **IEC** Immediate Eff.	1 2014 **DM** Degree Mass	10 2012 **LAPC** Laplacian C.	0 2012 **ABC** Attentive BC	1699 2001 **STRC** Straightness C	0 2015 **SNR** Silent Node R.	15 2011 **HPC** Harm. Prot.	26 2011 **LAC** Local Average	119 2008 **DMNC** DMNC	3 2013 **LR** Lurker Rank	2457 1987 **β-C** β Cent.	X X **HYP** Hyperbolic C.	27 2012 **kEPC** k-edge PC	13 2007 **FC** Functional C.	0 2014 **HCC** Hierar. CC
7	484 2005 **SC** Subgraph	613 1991 **FBC** Flow BC	14 2012 **RLBC** RLimited BC	477 1991 **MEC** Mediative Eff.	69 2010 **LEVC** Leverage Cent.	35 2010 **TC** Topological C.	X X **SDC** Sphere Degree	15 2010 **ZC** Zonal Cent.	14 2013 **CI** Collab. Index	11 2013 **CoEWC** CoEWC	45 2012 **NC** NC	108 2010 **MLC** Moduland C.	X X **RSC** Resolvent SC	1 2014 **SWIPD** SWIPD	36 2009 **XXXX** LinComb	0 2014 **BCPR** BCPR	0 2014 **TPC** Tunable PC	0 2015 **EDCC** Effective Dist.

citations year **C** Name

8000 1979 **Freeman** Conceptual	942 1966 **Sabidussi** Axiomatic	573 2006 **Borgatti/Everett** Conceptual	1130 2005 **Borgatti** Conceptual	24 2014 **Boldi/Vigna** Axiomatic	252 1974 **Nieminen** Axiomatic	6 1981 **Kishi** Axiomatic	3 2012 **Kitti** Axiomatic	3 2009 **Garg** Axiomatic

2065 1934 **Moreno** Historic	1546 1950 **Bavelas** Historic	780 1948 **Bavelas** Historic	1475 1951 **Leavitt** Historic	297 1992 **Borgatti/Everett** Conceptual	3649 2001 **Jeong et al.** Empirical	4167 1998 **Tsai/Ghoshal** Empirical	961 1993 **Ibarra** Empirical	71 2008 **Valente** Empirical

- 🟩 "Traditional"
- 🟪 Betweenness-like
- 🟩 Friedkin Measures
- 🟥 Miscellaneous
- 🟦 Path-based
- 🟧 Specific Network Type
- 🟩 Spectral-based
- 🟪 Closeness-like

Homework for next week / Upcoming lab session

- Make serious progress with Assignment 1
- Make choice of participation in course explicit. Un-enroll no later than September 25; anyone registered after that date will get a grade
- Consult the list of project topics on course website, and think of what you may want to work on
- Topic selection on Brigthspace opens Wednesday September 27 at 9:00; first come, first serve
- **Today:** stick around if you are already certain that you will take the course, and want to find a teammate already
- Next lab session: Friday September 29 from 9:00 to 10:45 in Snellius computer rooms
- Chance to ask final questions about Assignment 1