A short note on Hamiltonian circuits in subgraphs of the triangulation graph

A survey of results

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In this short note we consider the set $Tri(n)$ consisting of all triangulations of the regular n-gon ($n \geq 7$). $Tri(n)$ becomes a graph if we say that two triangulations are adjacent if and only if there is a "flip" transforming these triangulations into one another. A flip toggles the diagonal in one rectangle in the triangulation. For details, and the connection with rotations of binary trees, the reader is referred to [LUCAS]. The major problems are:

(A) Determine the diameter of $Tri(n)$.
(B) Examine the Hamiltonian circuits of $Tri(n)$ (they exist by a result of [LUCAS]).
(C) Find the shortest path between two given triangulations.

We now construct certain subgraphs of $Tri(n)$, and consider problem (B). Let k be an integer, $0 \leq k \leq \lfloor \frac{1}{2}(n-4) \rfloor$. Then $Tri(n,k)$ is the subgraph of $Tri(n)$ containing all triangulations of the regular n-gon having exactly k internal triangles. An internal triangle is a triangle that does not use any sides of the n-gon. We have:

LEMMA The number of elements of $Tri(n,k)$ is

$$n \binom{n-4}{2k} 2^{n-2k-4} \frac{1}{k+1} \binom{2k}{k} \frac{1}{k+2}.$$

Summation over k gives the Catalan number $\frac{1}{n-1}\binom{2(n-1)}{n-2}$, which is the number of elements of $Tri(n)$ (this follows by using the hypergeometric function ${}_2F_1$). Furthermore, there is —up to a factor $k+2$— an effective way to enumerate $Tri(n,k)$. If n is not equal to $2k+4$ then $Tri(n,k)$ is connected; notice that $Tri(2k+4,k)$ consists of two disjoint copies of $Tri(k+2)$.

THEOREM $Tri(n,0)$ has at least

$$cn2^{0.006n2^n}$$

Hamiltonian circuits, where c is an explicitly known constant.

We also have some results for small n.

[LUCAS] J.M. Lucas, The rotation graph of binary trees is Hamiltonian, *J. Algorithms* **8** (1987), 503–535.

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