Seminar (Combinatorial) Algorithms

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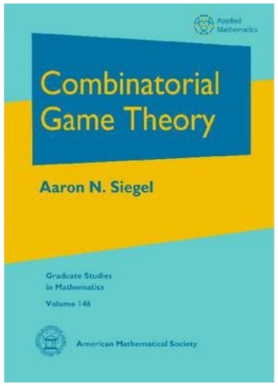
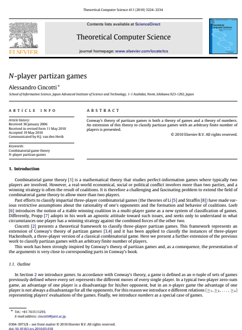
Spring 2018, Snellius 407–409

Tuesday 6.2.2018, 11:00–12:45

www.liacs.leidenuniv.nl/~kosterswa/semalg/

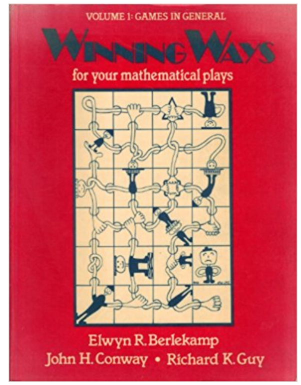
We discuss texts dealing with Combinatorial Games.

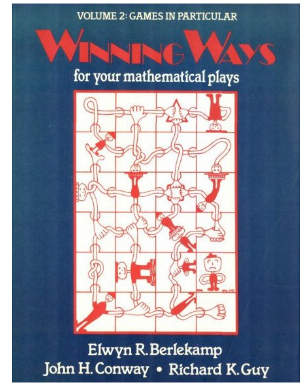


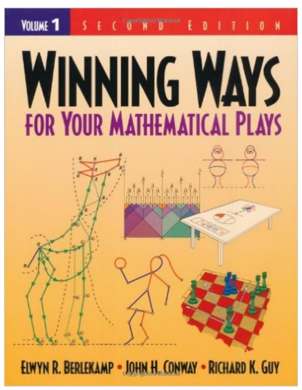

Theoretical Computer Science

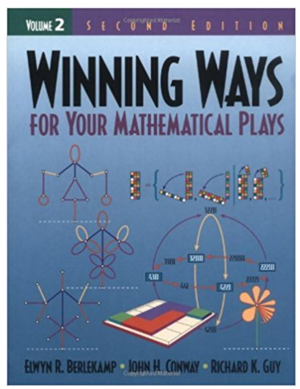
N-player partizan games

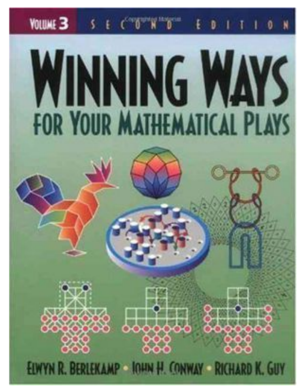
Alessandro Cincotti

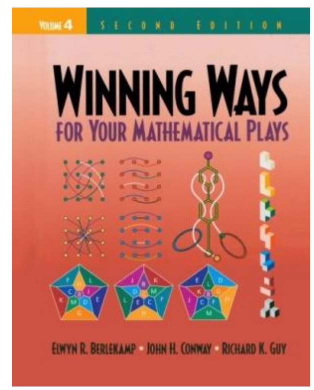












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First we examine two example games:

- Hackenbush

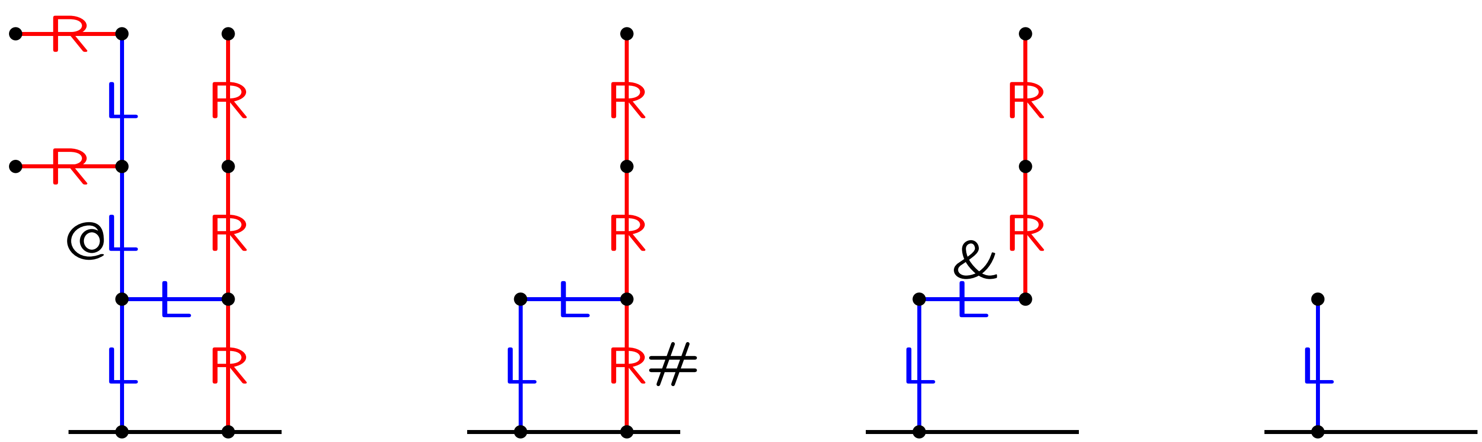
- Nim

And then:

- Literature

- How is the seminar organized?

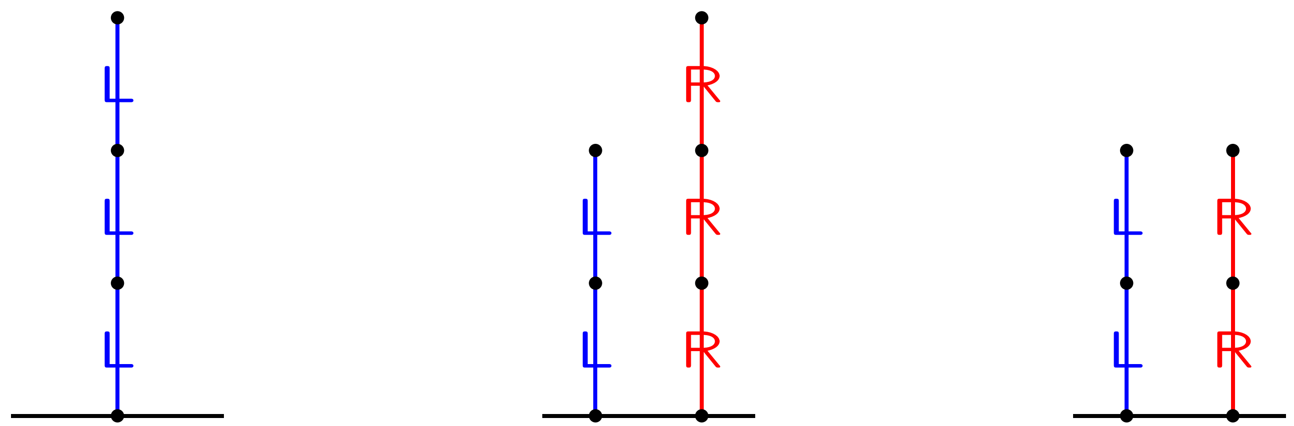
In the game (Blue-Red-)Hackenbush Left = she and Right = he alternately remove a bLue or a Red edge. All edges that are not connected to the ground, are also removed. He/she who removes the last edge wins!



Left chooses @, Right # (stupid), Left & and wins

BTW: here Right can always win, whoever begins!

What is the Hackenbush value of a position?



value 3 value $2-3=-1$ value $2-2=0$

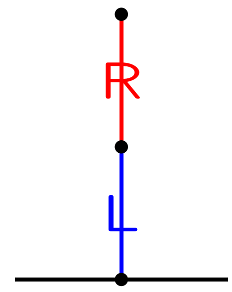
value >0: Left wins (whoever starts) $\mathcal{L}$

value $=0$: player to move loses $=$ first player loses $\mathcal{P}$

value <0: Right wins (whoever starts) $\mathcal{R}$

Remarkable: in this game no "player to move win"! $\mathcal{N}$ (*)

But what is the value of this position?



If Left starts, she wins immediately. If Right starts, Left can still move and wins. So Left always wins, and the value is > 0.

Question: is the value equal to 1?

If the value on the left would be 1, the value on the right would be $1 + (-1) = 0$, and the player to move should lose. Is this true?

No! If Left begins, Left loses, and if Right begins, Right can win. So Right always wins (= can always win), and the position on the right is < 0, and the position on the left is between 0 and 1.

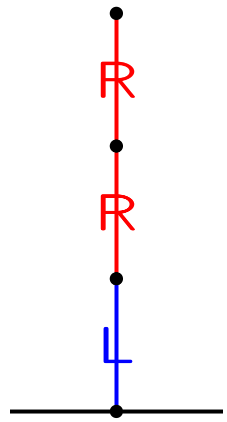
The position on the left is denoted $\{\, 0 \mid 1 \,\}$.

The position on the right has value 0: the player to move loses. And so we have

$$\{\, 0 \mid 1 \,\} + \{\, 0 \mid 1 \,\} + (-1) = 0,$$

and “apparently” $\{\, 0 \mid 1 \,\} = 1/2$.

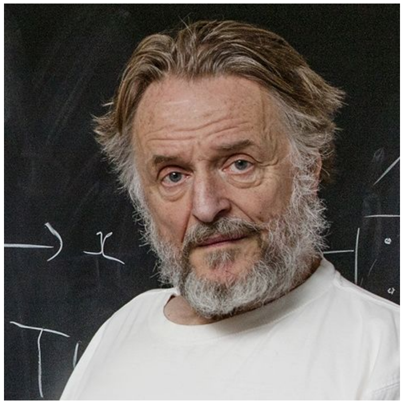
We denote the value of a position where Left can play to (values of) positions from the set L and Right can play to (values of) positions from the set R with $\{ L \mid R \}$.
An example:



The value here is $\{ 0 \mid \frac{1}{2}, 1 \} = \frac{1}{4}$.

Simplicity rule: The value is always the "simplest" number between left and right set, i.e., the smallest integer — or the dyadic number with the lowest denominator (power of 2).

Knuth and Conway



Donald E.(Ervin) Knuth
1938, US
NP; KMP
TEX
change-ringing; 3:16
The Art of Computer
Programming

John H.(Horton) Conway
1937, UK $\rightarrow$ US
Co_1, Co_2, Co_3
Doomsday algoritme
game of Life; Angel problem
Winning Ways for your
Mathematical Plays

Surreal numbers

In this way we define surreal numbers: “decent” pairs of sets of previously defined surreal numbers: all elements from the left set are smaller than those from the right set.

We start with $0 = \{ \emptyset \mid \emptyset \} = \{ \text{NOTHING} \mid \text{NOTHING} \} = \{ \mid \}$: the game where the player to move does not have any move, and loses: born on day 0.

And then $1 = \{ 0 \mid \}$ en $-1 = \{ \mid 0 \}$, born on day 1.
And $42 = \{ 41 \mid \}$, born on day 42.
And $\frac{3}{8} = \{ \frac{1}{4} \mid \frac{1}{2} \}$, born on day 4.
And $\pi = \{ 3, 3\frac{1}{8}, 3\frac{9}{64}, \ldots \mid 4, 3\frac{1}{2}, 3\frac{1}{4}, 3\frac{3}{16}, 3\frac{5}{32}, \ldots \}$.

We define, e.g.:

$$\varepsilon = \{\, 0 \mid \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \,\},$$

an "incredibly small number", and

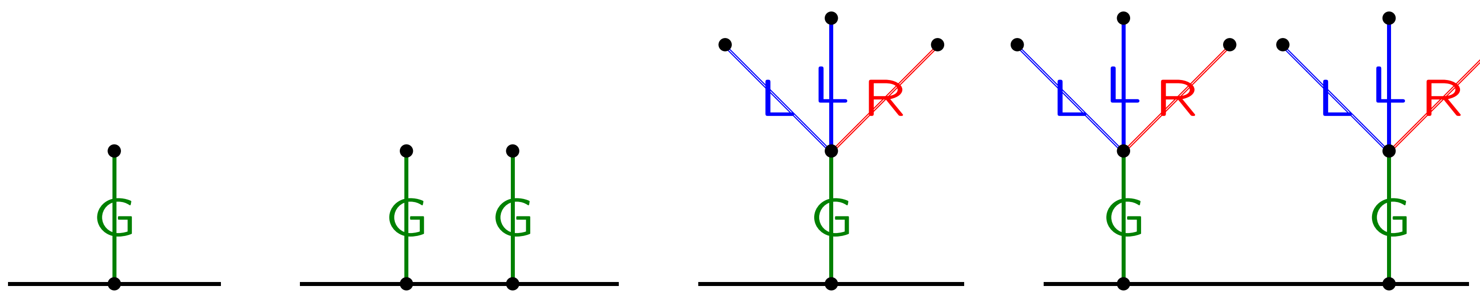
$$\omega = \{\, 0, 1, 2, 3, \ldots \mid \,\} = \{\, \mathbf{N} \mid \emptyset \,\},$$

a "terribly large number, some sort of ∞".

Then we have $\varepsilon \cdot \omega = 1$ — if you know how to multiply.

And then $\omega + 1$, $\sqrt{\omega}$, ω^{ω}, $\varepsilon/2$, and so on!

In Red-Green-Blue-Hackenbush we also have Green edges, that can be removed by both players.



The first position has value $* = \{\ 0 \mid 0\ \}$ (not a surreal number), because the player to move can win.

The second position is $* + * = 0$ (player to move loses).

The third position is a first player win.

The fourth position is a win for Left (whoever begins), and is therefore > 0.

How to add surreal numbers? Like this:

$$a + b = \{\, A_L + b, a + B_L \mid A_R + b, a + B_R \,\}$$

if $a = \{\, A_L \mid A_R \,\}$ and $b = \{\, B_L \mid B_R \,\}$.
Here we put $u + \emptyset = \emptyset$ and $u + V = \{u + v : v \in V\}$.

This corresponds with the following: you play two games in parallel, and in every move you must play in exactly one game: the disjunctive sum.

Exercise: verify that

$$1 + \frac{1}{2} = \{\, 1 \mid 2 \,\} = \frac{3}{2}\,.$$

Reference: Claus Tøndering's paper

In the game of Nim you have piles/heaps of coins/matches, and in every move a player must take any number of coins/matches from one pile/heap.

Again: the player who cannot move loses: normal play.

Bouton's theorem from 1901 says: The Nim position with heaps of sizes $a_1, a_2, \ldots, a_k$ is a first player loss ($\mathcal{P}$-position) if and only if $a_1 \oplus a_2 \oplus \ldots \oplus a_k = 0$ (see next slide). Otherwise it is a first-player win ($\mathcal{N}$-position).

Nim is an impartial game: players have the same moves.

We define the nim-sum $x \oplus y$ as the binary sum "without carry" of x and y.

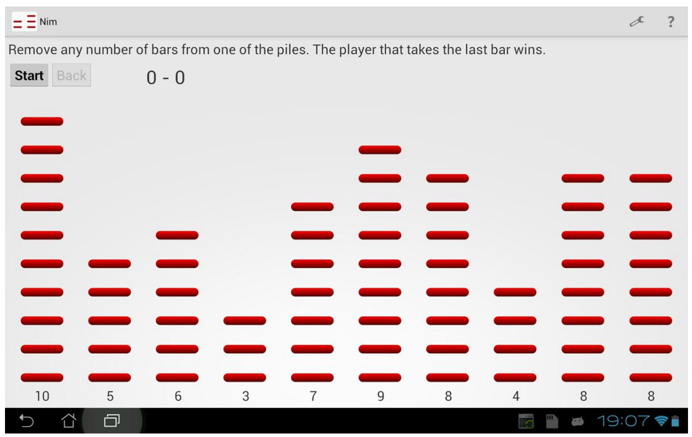
So what about Nim with piles of sizes 3, 5 and 8?

We compute $3 \oplus 5 \oplus 8 = 7 \neq 0$, since $0011 \oplus 0101 \oplus 1000 = 1110$. So it is a first person win, and a winning move (in this case unique) leads to 0: remove two objects from the third pile $8 \to 6$.

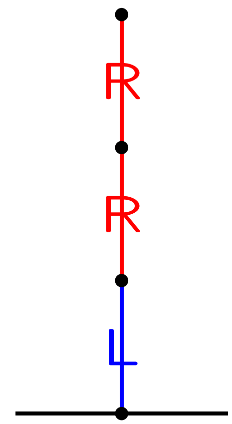
A Nim position with a single pile of m tokens has game value $*m$ (the same as a stalk of m green Hackenbush edges; not a surreal number). Here m is the nim value, and the Sprague-Grundy theorem states that every position in a “short” impartial game is “equal” to such a nim-heap.

Suppose we have a game where we can choose between a game of Nim with value $*m$ and one with value $*n$. Then its value is $*\text{mex}(m, n)$, where $\text{mex}(m, n) =$ the smallest integer ≥ 0 that differs from m and n, the so-called minimal excluded value.

Now consider this addition of two game positions, with on the left a Nim position and on the right a Hackenbush position:



$+$



Then this sum is > 0, it is a win for left! In general: $*m + 1/1024 > 0$.

Two main references:

Siegel:
A.N. Siegel, Combinatorial Game Theory, AMS, 2013.

WinningWays:
E.R. Berlekamp, J.H. Conway and R.K. Guy, Winning Ways for your Mathematical Plays, 1982/2001.
(Note that there are two editions: the first has two volumes, the second has four volumes. Page numbers below refer to the second edition, and differ a little from those of the first edition. In all cases: volume 1.)

And the subjects (prerequisites mentioned in [. . .]):

1. Hackenbush, Siegel, pp. 15–21; WinningWays, pp. 2–7.
2. Redwood furniture, WinningWays, pp. 211–214. [Hackenbush]
3. Cutcake and Maundy Cake, WinningWays, pp. 24–27. Also Ski-jumps?
4. Sprague-Grundy, Siegel, pp. 177–183;
   Wikipedia: en.wikipedia.org/wiki/Sprague_Grundy_theorem.
5. Heap games (including Octal games), Siegel, pp. 184–192.
6. The group G, Siegel, pp. 53–63.
7. Infinitesimals A, Siegel, pp. 82–97.
8. Infinitesimals B, Siegel, pp. 82–97. [Infinitesimals A]

TODO: Simplicity theorem, Clobber, Toads and frogs, Domineering.

www.liacs.leidenuniv.nl/~kosterswa/semalg/subjects.pdf

How is the seminar organized? Do the following twice:

> Present a (chosen) "paper" during a 45 minutes lecture. Make slides, and use the blackboard.
>
> Produce a 7–10 page paper/report in LaTeX/PDF. Use your own words, no copy-paste; English.

Grading is based on the four Ps: presentation (2×), paper (2×), participation (including presence: discussions, questions) and maybe peer review OR programming.

Apply for participation: send e-mail† with proof of (*) from slide 5 before Monday afternoon February 12, 2018. At most ≈ 10 participants.

† `w.a.kosters@liacs.leidenuniv.nl`