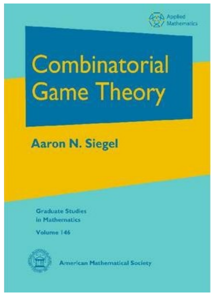
Game Complexity

Combinatorial Game Theory



Applied Mathematics

Combinatorial
Game Theory

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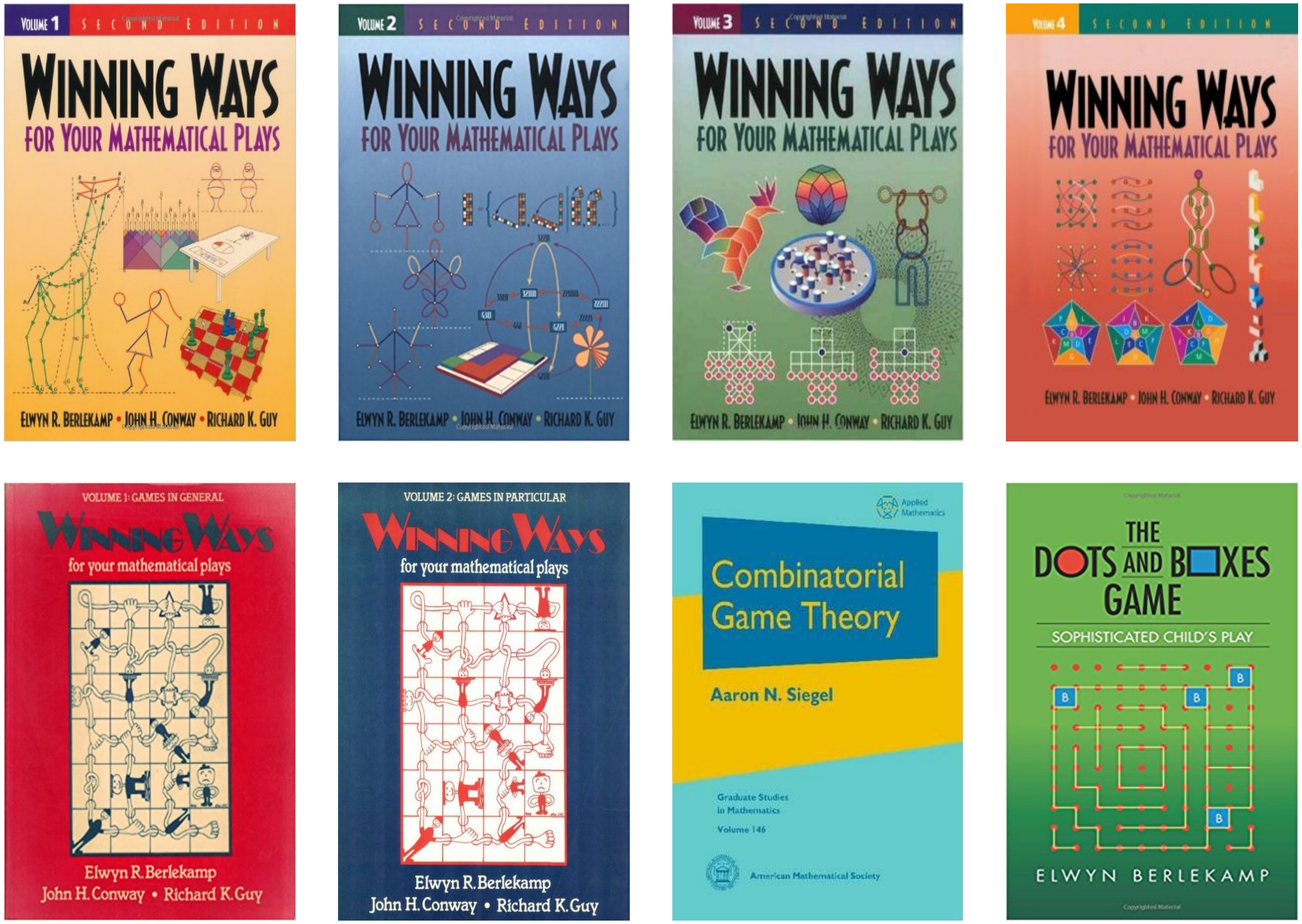
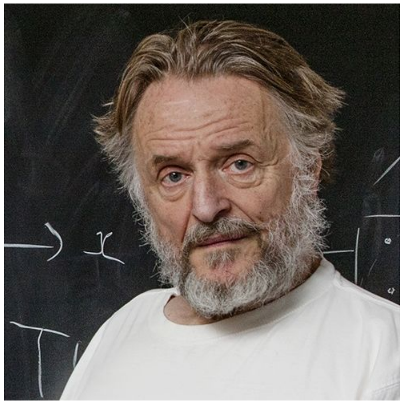
Winning ways



Combinatorial Game Theory

Berlekamp, Conway & Guy: Winning ways

Knuth and Conway



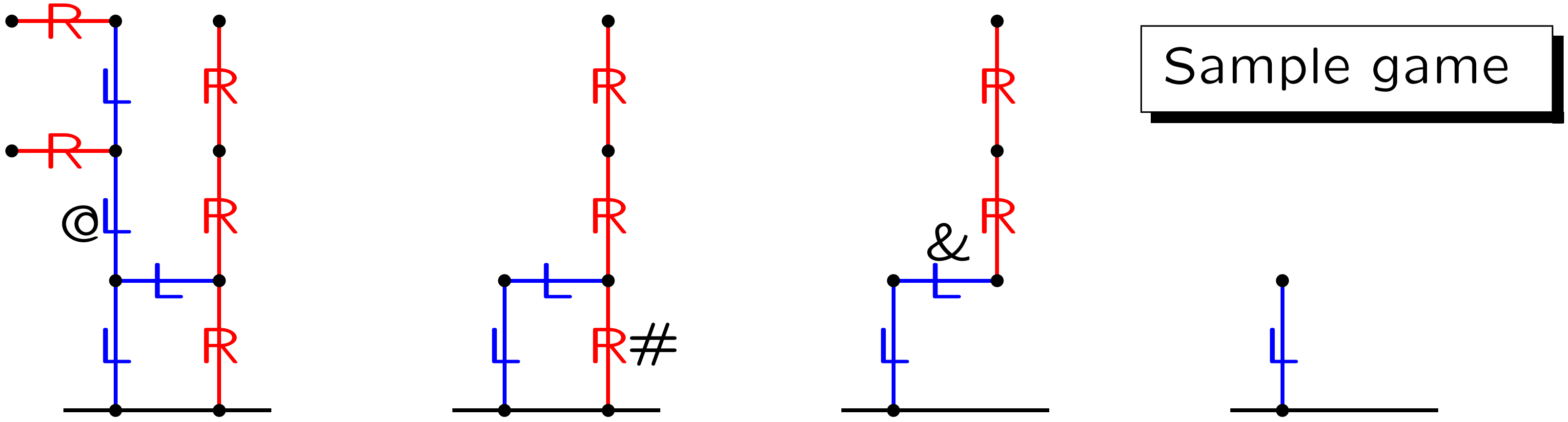
Donald E.(Ervin) Knuth
1938, US
NP; KMP
TEX
change-ringing; 3:16
The Art of Computer
Programming

John H.(Horton) Conway
1937, UK → US
Co_1, Co_2, Co_3
Doomsday algoritme
game of Life; Angel problem
Winning Ways for your
Mathematical Plays

Surreal numbers

Hackenbush

In the game (Blue-Red-)Hackenbush Left = she and Right = he alternately remove a bLue or a Red edge. All edges that are no longer connected to the ground, are also removed. *If you cannot move, you lose!*

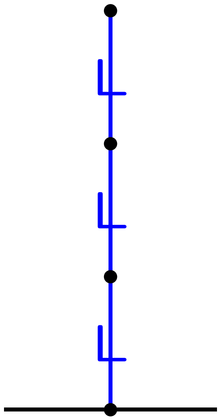


Left chooses @, Right # (stupid), Left &. Now Left wins because Right cannot move.

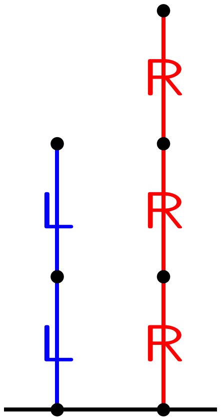
By the way, Right can win here, whoever starts!

Value of a position

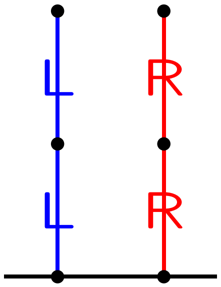
When playing Hackenbush, what is the value of a position?



value 3



value $2 - 3 = -1$



value $2 - 2 = 0$

value > 0: Left wins (whoever starts) $\mathcal{L}$

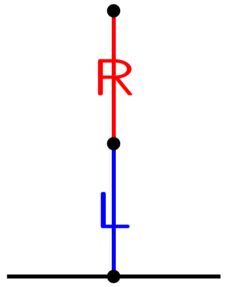
value $= 0$: first player loses $\mathcal{P}$

value < 0: Right wins (whoever starts) $\mathcal{R}$

Remarkable: Hackenbush has no “first player wins”! $\mathcal{N}$

For what it's worth

But what is the value of this position??



If Left begins, she wins immediately. If Right begins, Left can still move, and also wins. So Left always wins. Therefore, the value is > 0.

Exercise: $\boxed{\text{Is the value equal to 1?}}$

Perhaps 1?

If the value in the left hand side position would be 1, the value of the right hand side position would be $1+(-1)=0$, and the first player should lose. Is this true?

No! If Left begins, Left loses, and if Right begins Right can also win. So Right always wins (i.e., can always win), and therefore the right hand side position is < 0, and the left one is between 0 and 1.

1/2

The left hand side position is denoted by $\{\, 0 \mid 1 \,\}$.

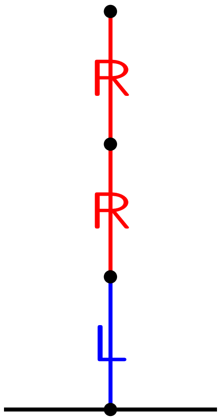
Note that the right hand side position does have value 0: the first player loses. And so we have:

$$\{\, 0 \mid 1 \,\} + \{\, 0 \mid 1 \,\} + (-1) = 0,$$

and "apparently" $\{\, 0 \mid 1 \,\} = 1/2$.

More general

We denote the value of a position where Left can play to (values of) positions from the set L and Right can play to (values of) positions from the set R by $\{ L \mid R \}$.



Its value is $\{ 0 \mid \frac{1}{2}, 1 \} = \frac{1}{4}$.

Simplicity rule: The value is always the "simplest" number between left and right set: the smallest integer — or the dyadic number with the lowest denominator (power of 2).

Exercise: Give a position with value 3/8.

Exercise: Show that $\{ 0 \mid 100 \} = 1$.

Surreal numbers

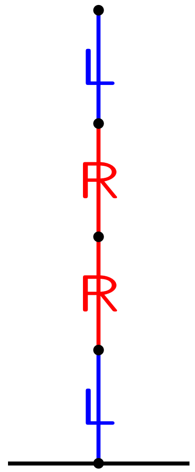
In this way we define surreal numbers: “decent” pairs of sets of previously defined surreal numbers: all elements from the left set are smaller than those from the right set.

Start with $0 = \{\, \emptyset \mid \emptyset \,\} = \{\, \text{nothing} \mid \text{nothing} \,\} = \{\, \mid \,\}$: the game where both Left and Right have no moves at all, and therefore the first player loses: born on day 0.

And then $1 = \{\, 0 \mid \,\}$ and $-1 = \{\, \mid 0 \,\}$, born on day 1.

And $42 = \{\, 41 \mid \,\}$, born on day 42.

And $\frac{3}{8} = \{\, \frac{1}{4} \mid \frac{1}{2} \,\}$, born on day 4.



Wow!

Sets can be infinite: $\pi = \{\, 3, 3\frac{1}{8}, 3\frac{9}{64}, \ldots \mid 4, 3\frac{1}{2}, 3\frac{1}{4}, 3\frac{3}{16}, \ldots \,\}$.

We define, e.g.:

$$\varepsilon = \{\, 0 \mid \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \,\},$$

an "incredibly small number", and

$$\omega = \{\, 0, 1, 2, 3, \ldots \mid \,\} = \{\, \mathbb{N} \mid \emptyset \,\},$$

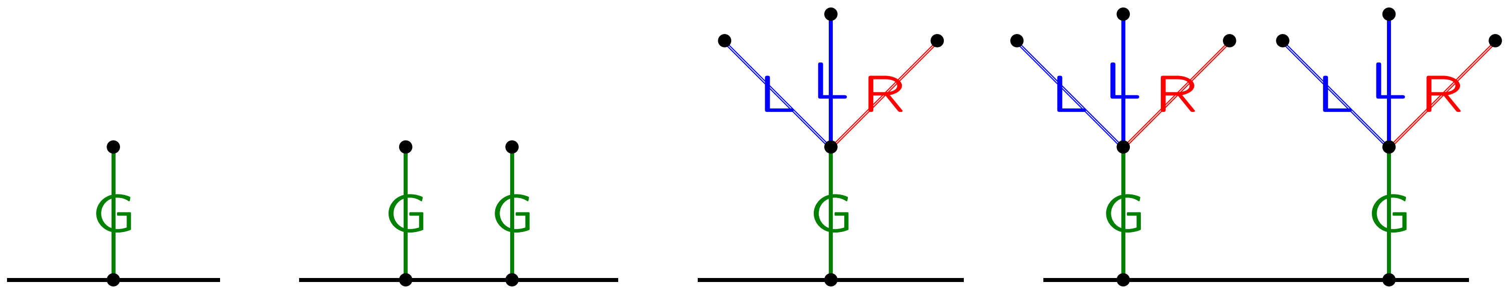
a "terribly large number, some sort of ∞".

Then we have $\varepsilon \cdot \omega = 1$ — if you know how to multiply.

And then $\omega + 1$, $\sqrt{\omega}$, ω^{ω}, $\varepsilon/2$, and so on!

Red-Green-Blue-Hackenbush

In Red-Green-Blue-Hackenbush we also have Green edges, that can be removed by both players.



The first position has value $* = \{ 0 \mid 0 \}$ (not a surreal number): a first player win.

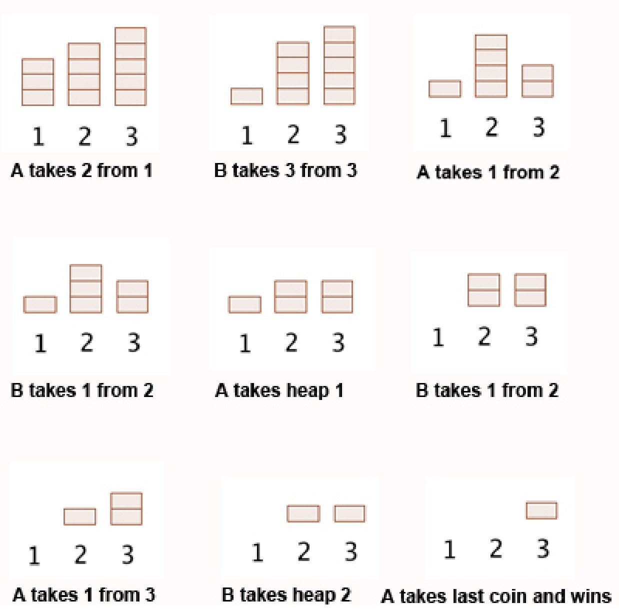
The second position is $* + * = 0$ (first player loses).

The third position is again a first player win.

The fourth position is a win for Left (whoever begins), and is therefore > 0.

Nim

In the Nim game we have several stacks of tokens = coins = matches. A move consists of taking a nonzero number of tokens from one of the stacks. If you cannot move, you lose ("normal play").



The game is impartial: both players have the same moves. (And for the misère version: if you cannot move, you win.)

Nim — analysis

For Nim we have Bouton's analysis from 1901.

We define the nim-sum $x \oplus y$ of two positive integers x en y as the bitwise XOR of their binary representions: addition without carry. With two stacks of equal size the first player loses ($x \oplus x = 0$): use the "mirror strategy".

A nim game with stacks of sizes $a_1, a_2, \ldots, a_k$ is a first player loss exactly if $a_1 \oplus a_2 \oplus \ldots \oplus a_k = 0$. And this sum is the Sprague-Grundy value.

We denote a game with value m by $*m$ (the same as a stalk of m green Hackenbush edges; not a surreal number). And $*1 = *$. So if $m \neq 0$ the first player loses.

The Sprague-Grundy Theorem roughly says that every impartial gave is a Nim game.

With stacks of sizes 29, 21 and 11, we get $29 \oplus 21 \oplus 11 = 3$:

```
11101      29
10101      21
 1011      11
-----      --
00011       3
```

So a first player win, with unique winning move $11 \to 8$.

Exercise: Why this move, and why is it unique?

How to add these "games" (we already did)? Like this:

$$a + b = \{\, A_L + b, a + B_L \mid A_R + b, a + B_R \,\}$$

if $a = \{\, A_L \mid A_R \,\}$ and $b = \{\, B_L \mid B_R \,\}$.
Here we put $u + \emptyset = \emptyset$ and $u + V = \{u + v : v \in V\}$.

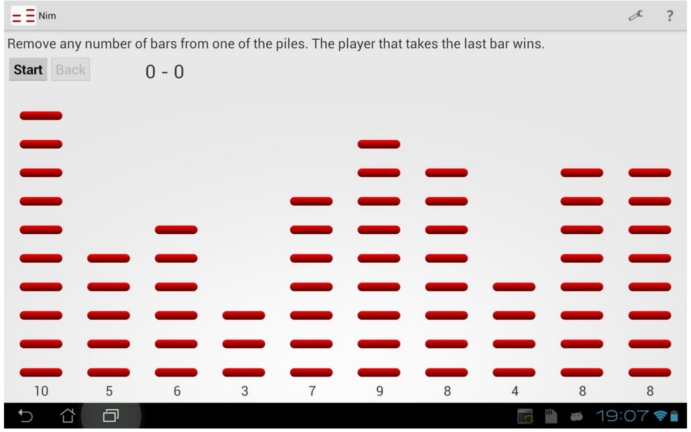
This corresponds with the following: you play two games in parallel, and in every move you must play in exactly one game: the disjunctive sum.

Exercise: $\boxed{\text{Verify that } 1 + \frac{1}{2} = \{\, 1 \mid 2 \,\} = \frac{3}{2}\,.}$

See Claus Tøndering's paper

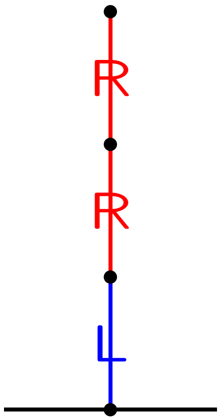
Example addition

Consider this addition of two game positions, on the left a Nim position and on the right a Hackenbush position:



Remove any number of bars from one of the piles. The player that takes the last bar wins.

$+$



Then this sum is > 0, it is a win for Left! More general: $*m + 1/1024 > 0$.

Exercise: $\boxed{\text{Prove this.}}$

Clobber

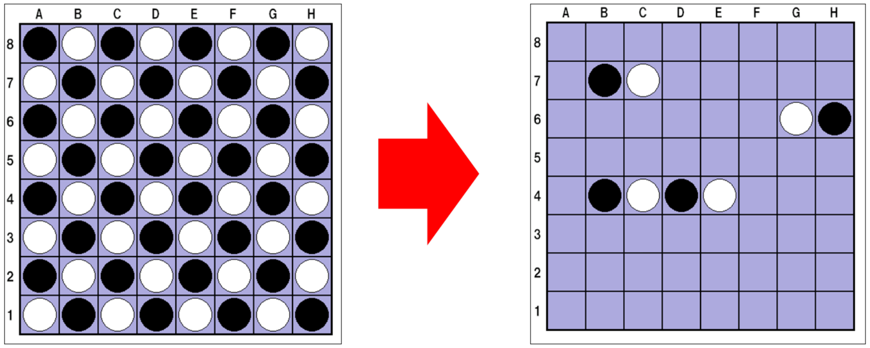
We finally play Clobber, on an m times n board, with white (Right) and black (Left) stones. A stone can capture = "clobber" a directly adjacent stone from the other color. If you cannot move, you lose.

Some examples:

●○ $= \{ 0 \mid 0 \} = *$

●●○ $= \{ 0 \mid * \} = \uparrow > 0$

●○●○●○ $= 0$



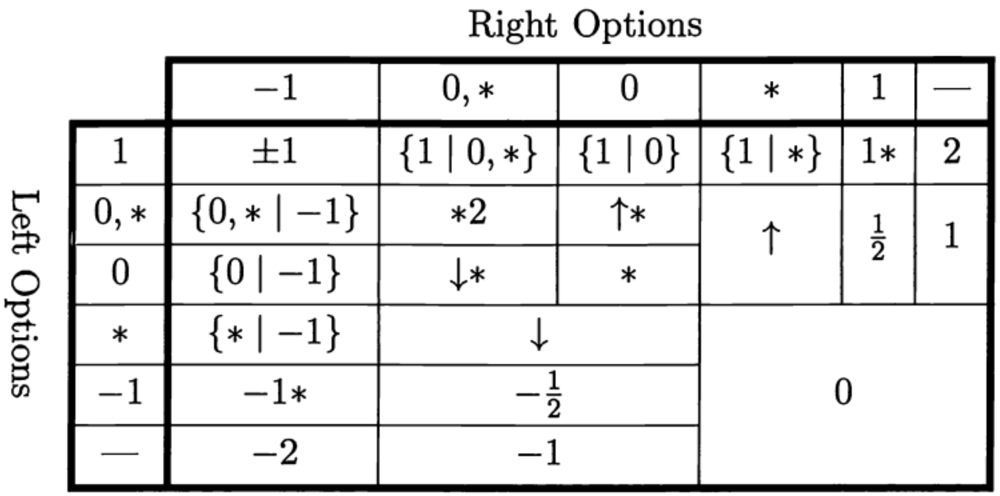
●○●○●○●○●○ $= \pm(\uparrow, \uparrow^{[2]}*, \{0|\uparrow, \pm(*,\uparrow)\}, \{\Uparrow*|\downarrow, \pm(*,\uparrow)\})$

Exercise: $\boxed{\text{Show that } \uparrow < 1/2^n \text{ for all } n > 0.}$

Simple games

From the Siegel book:

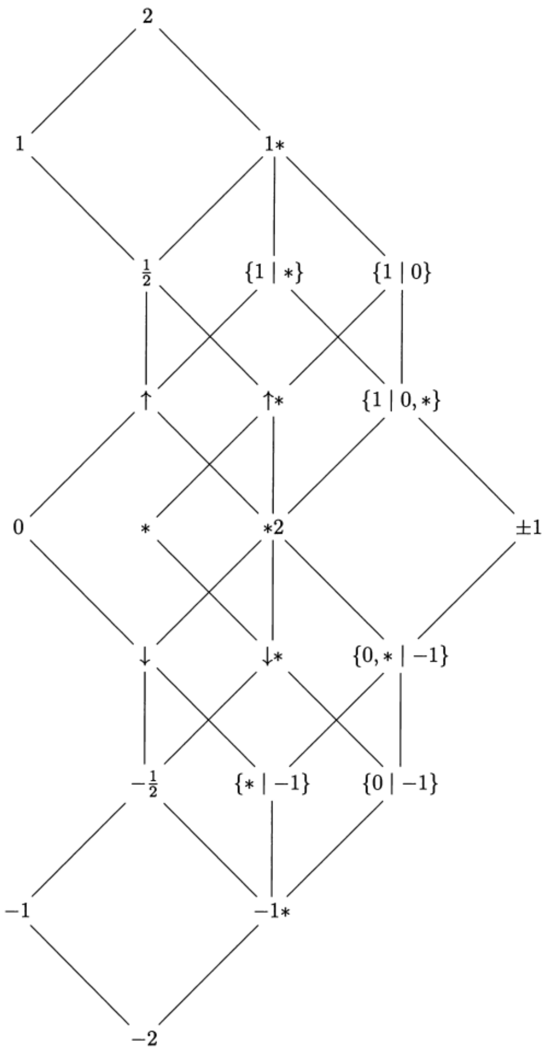
$0 < \{\ 0 \mid * \ \} = \uparrow < 1/2 < \{\ 1 \mid 1\ \} = 1* < 2$



Right Options

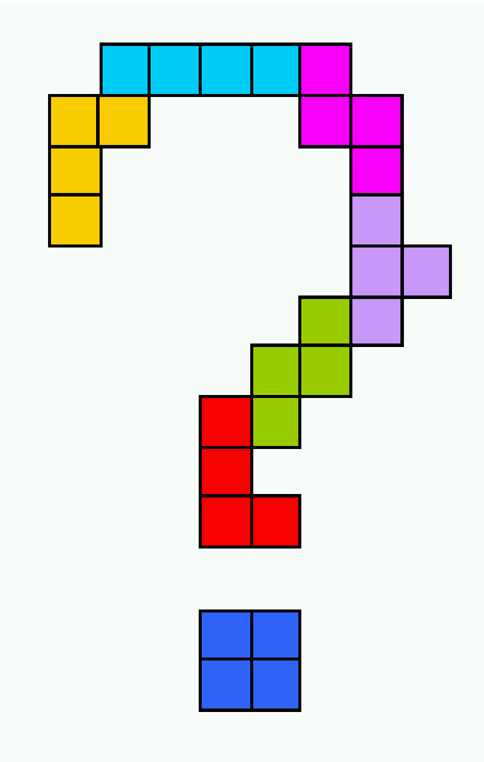
Left Options

	-1	$0, *$	0	$*$	1	—
1	± 1	$\{1 \mid 0, *\}$	$\{1 \mid 0\}$	$\{1 \mid *\}$	$1*$	2
$0, *$	$\{0, * \mid -1\}$	$*2$	$\uparrow*$	$\uparrow$	$\frac{1}{2}$	1
0	$\{0 \mid -1\}$	$\downarrow*$	$*$			
$*$	$\{* \mid -1\}$	$\downarrow$		0		
-1	$-1*$	$-\frac{1}{2}$				
—	-2	-1				



“?”

Two-player games with no hidden information and no chance elements have a complex and beautiful structure.
See A.N. Siegel, Combinatorial Game Theory, 2013.



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