Foundations of Computer Science

Fundamentele Informatica 1

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Leiden Institute of
Advanced Computer Science

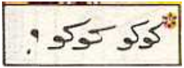
Hoofdstuk 3

Functies

$\text{Functions } \underbrace{\text{and Algorithms}}_{\text{andere vakken}}$

3 Functies

- Begrippen
- Injectief, surjectief
- Cardinaliteit
- Rijen



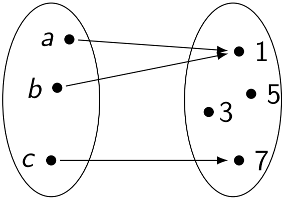
informeel

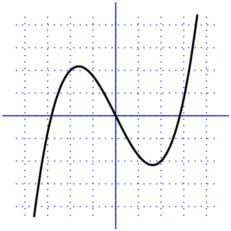
A, B verzamelingen

functie $f : A \to B$ afbeelding, *map*

definitie

een *voorschrift* dat voor elk element x in A één element $y \in B$ toevoegt





$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, \ldots$

functie $f : A \to B$

grafiek van f $\quad \{ (x, f(x)) \mid x \in A \} \subseteq A \times B$ $\quad$ binaire relatie

$$f(x) = 2(x+1) \qquad\qquad g(x) = 2x+2$$

$$h(x) = (x{+}1)^2 - (x{+}3)(x{-}2) + (x{-}5)$$

dezelfde uitkomsten

$f, g : A \to B$ $\quad$ *gelijk*

$f = g$ $\quad$ desda $\quad$ $f(a) = g(a)$, voor alle $a \in A$

(informeel) voorschrift $\rightsquigarrow$ grafiek $\rightsquigarrow$ binaire relatie (formeel)

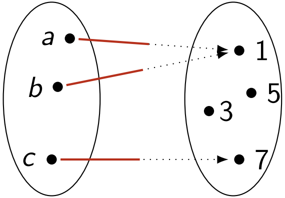
Definitie

$R \subseteq A \times B$ *functie* voor elke $x \in A$ precies één $y \in B$ zodat xRy

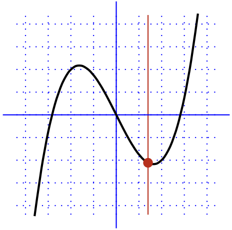
ten minste één totaal

ten hoogste één functioneel

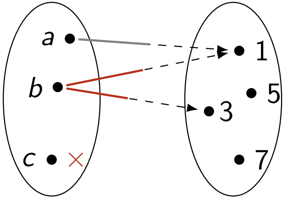
$R \subseteq A \times B$ *functie* totaal & functioneel *functie*



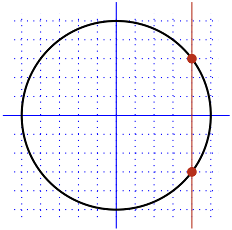
	1	2	3	4
a	1	0	0	0
b	0	0	1	0
c	0	0	1	0



maar niet . . .

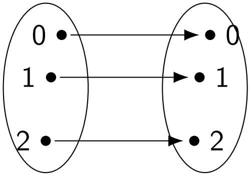


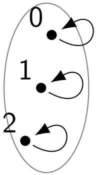
	1	2	3	4
a	1	0	0	0
b	0	1	1	0
c	0	0	0	0

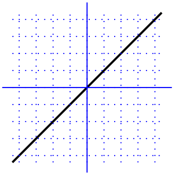


nu als functie

identiteit id_A 1_A $\{\, (a,a) \mid a \in A \,\} \subseteq A \times A$





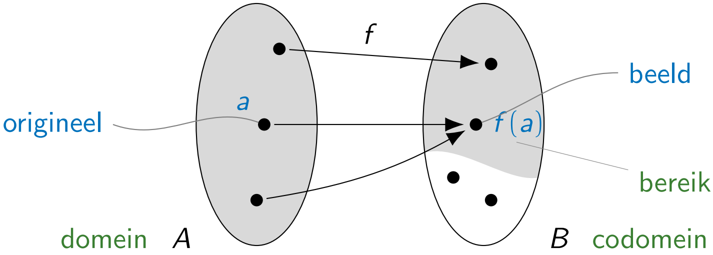


$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

$$f : A \to B$$

$$y = f(x)$$

$$x \mapsto f(x)$$



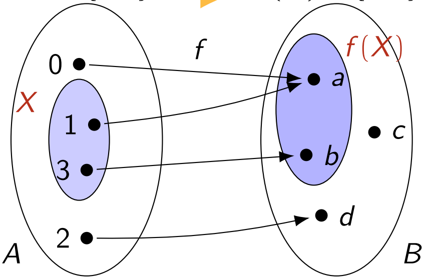
$f : A \to B$

$X \subseteq A$ *beeld* $f(X) = \{ f(x) \mid x \in X \}$

$= \{y \in B \mid y = f(x) \text{ voor zekere } x \in X\}$

$f : \{0, 1, 2, 3\} \to \{a, b, c, d\}$

$X = \{1, 3\}$ $f(X) = \{a, b\}$



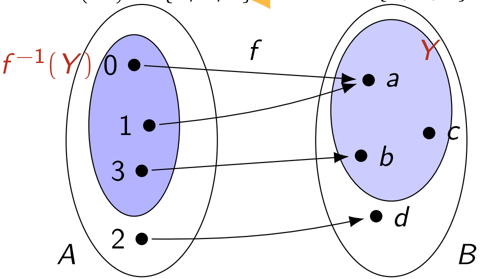
$f : A \to B$

$Y \subseteq B$ *origineel* $f^{-1}(Y) = \{\, x \in A \mid f(x) \in Y \,\}$

$= \{x \in A \mid f(x) = y \text{ voor zekere } y \in Y\}$

$f : \{0, 1, 2, 3\} \to \{a, b, c, d\}$

$f^{-1}(Y) = \{0, 1, 3\}$ $Y = \{a, b, c\}$



$$f : \{0,1,2,3\} \to \{a,b,c,d\}$$

$X = \{1,3\}$

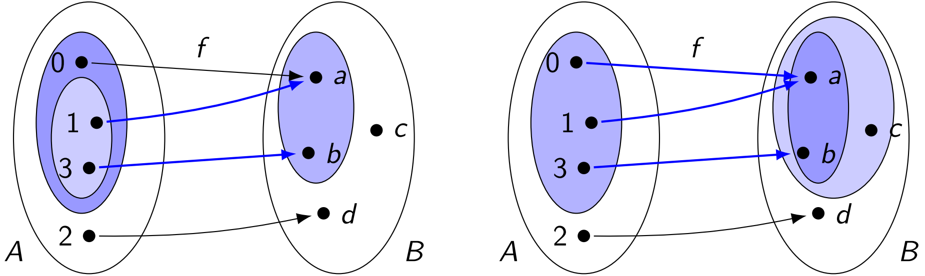
$f(X) = \{a, b\}$

$f^{-1}(f(X)) = \{0,1,3\}$

$Y = \{a,b,c\}$

$f^{-1}(Y) = \{0,1,3\}$

$f(f^{-1}(Y)) = \{a,b\}$



Zij $f : A \to B$ een functie.

(i) Voor elke $X \subseteq A$ geldt: $f^{-1}(f(X)) \supseteq X$.

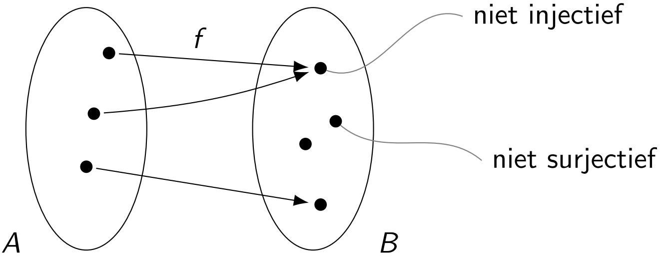
(ii) Voor elke $Y \subseteq B$ geldt: $f(f^{-1}(Y)) \subseteq Y$.

3 Functies

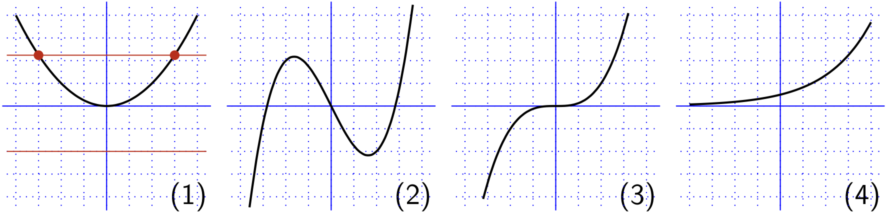
- Begrippen
- Injectief, surjectief
- Cardinaliteit
- Rijen

$f : A \to B$

- totaal $\mathrm{dom}(R) = A$
- *surjectief* $\mathrm{ran}(R) = B$ (of ‘*op*’)
- functioneel uit $x\,R\,y$ en $x\,R\,z$ volgt dat $y = z$
- *injectief* uit $f(x) = f(y)$ volgt dat $x = y$ (of *één-één*)

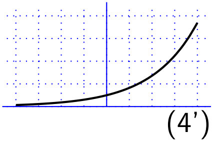


$f : \mathbb{R} \to \mathbb{R}$



(4) $\quad f : \mathbb{R} \to \mathbb{R} \quad x \mapsto e^x$
niet surjectief

(4') $\quad f' : \mathbb{R} \to \mathbb{R}^+ \quad x \mapsto e^x$
surjectief (en injectief)



surjectief injectief

	1	2	3	4
a	1	0	0	0
b	0	1	0	0
c	0	0	1	0

injectief

	a	b	c
1	1	0	0
2	0	1	0
3	0	1	0
4	0	0	1

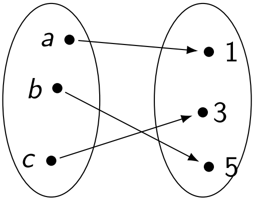
surjectief

	a	b	c
a	1	0	0
b	0	0	1
c	0	1	0

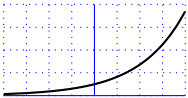
sur-/injectief

$f : A \to B$

bijectie surjectief & injectief op & 1-1

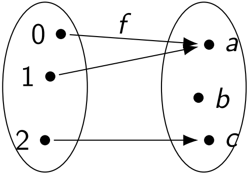


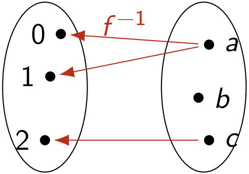
	naar			
van	0	1	2	3
0	1	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	1	0	0

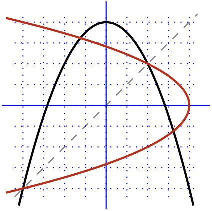


$f : \mathbb{R} \to \mathbb{R}^+$

$f : A \to B$ $\quad$ $f^{-1} : B \to A$ $\quad$ $\{ (y, x) \mid y = f(x) \}$



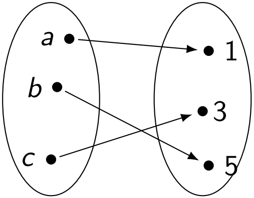




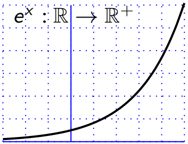
Thm. 3.1

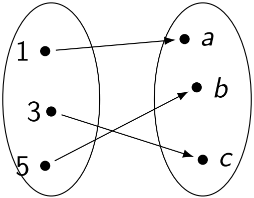
inverse functie f^{-1} van $f : A \to B$ bestaat desda f een bijectie is

bijectie $\quad$ surjectief & injectief $\quad$ precies één pijl aankomst

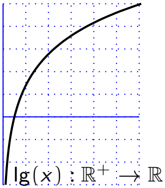


	0	1	2	3
0	1	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	1	0	0



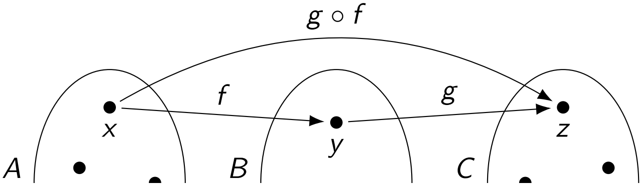


	0	1	2	3
0	1	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	1	0	0



samenstelling

$$h(x) = \sin(x + \tfrac{\pi}{2}) \qquad f(x) = x + \tfrac{\pi}{2} \qquad g(y) = \sin(y)$$



$$f : A \to B \qquad g : B \to C$$

$$f(x) = y \qquad g(y) = z$$

$$(g \circ f)(x) = g(f(x)) = g(y) = z$$

samenstelling (*compositie*) van f en g $\quad g \circ f : A \to C$

$$x \mapsto g(f(x))$$

samenstelling is associatief $(h \circ g) \circ f = h \circ (g \circ f)$

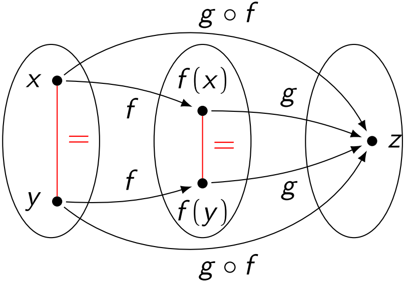
$f : A \to B \quad g : B \to C$

Prb. 3.7

als f, g *surjectief* 'op', dan ook $g \circ f$ minimaal één

als f, g *injectief* '1-1', dan ook $g \circ f$ maximaal één

h injectief $\quad h(x) = h(y) \quad$ dan $\quad x = y$



$(g \circ f)(x) = (g \circ f)(y) \quad$ dan $\quad x = y$

$g(f(x)) = g(f(y)) \quad \underbrace{\text{dan}}_{g} \quad f(x) = f(y) \quad \underbrace{\text{dan}}_{f} \quad x = y$

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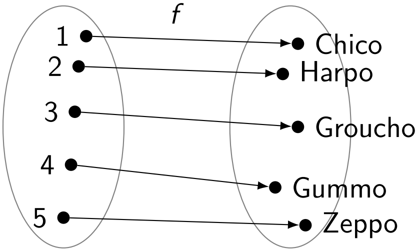
V eindig $\quad |V| = n$

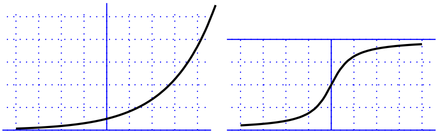
$f : \{1, 2, \ldots, n\} \to V \quad$ bijectie



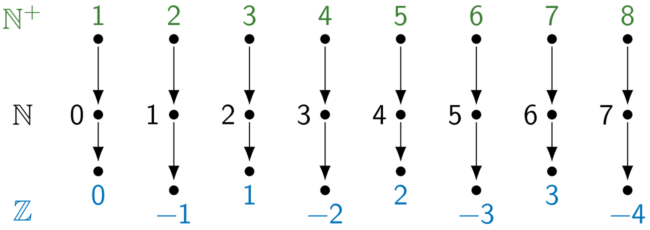
gelijkmachtig $A \simeq B$ bijectie $f : A \to B$

equipotent

$\mathbb{R} \simeq \mathbb{R}^+$ $\mathbb{R} \simeq (0,1)$



$\mathbb{N}^+ \simeq \mathbb{N} \simeq \mathbb{Z}$



gelijkmachtig $A \simeq B$ bijectie $f : A \to B$

equivalentierelatie

– $A \simeq A$ reflexief id : $A \to A$ indentiteit

– als $A \simeq B$ dan $B \simeq A$ symmetrisch

$f : A \to B$ dan $f^{-1} : B \to A$ inverse

– als $A \simeq B$ en $B \simeq C$ dan $A \simeq C$ transitief

$f : A \to B$, $g : B \to C$ dan $g \circ f : A \to C$ compositie

Sch 3.7 Cardinality (later behandeld!)

3 Functies

- Begrippen
- Injectief, surjectief
- Cardinaliteit
- Rijen

- $1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 9, \ldots$
- $\heartsuit, \spadesuit, \heartsuit, \clubsuit, \heartsuit, \diamondsuit, \diamondsuit, \ldots$

rij $\quad a_1, a_2, a_3, \ldots \qquad a : \mathbb{N}^+ \to A \qquad a : \mathbb{N} \to A$

a_n ipv $a(n)$

$(a_n)_{n\in\mathbb{N}} \qquad \{a_n\}_{n\in\mathbb{N}}$

- $\underset{1^e}{\frac{1}{1}}, \underset{2^e}{\frac{1}{2}}, \frac{1}{3}, \frac{1}{4}, \ldots \qquad \left(\frac{1}{n}\right)_{n\geqslant 1}$
- $\underset{0^e}{1}, \underset{1^e}{2}, \underset{2^e}{4}, 8, 16, \ldots \qquad (2^n)_{n\geqslant 0}$
- $\underset{0^e}{1}, \underset{1^e}{-1}, \underset{2^e}{1}, -1, 1, \ldots \qquad a_n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ oneven} \end{cases} \qquad (n \geqslant 0)$

$$a_n = (-1)^n$$

$$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$ *for loop*

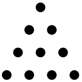
$$\textstyle\sum_{i=1}^{n} a_i = \sum_{i=1}^{n-1} a_i + a_n$$
$$\textstyle\sum_{i=1}^{1} a_i = a_1 \qquad \sum_{i=1}^{0} a_i = 0 \qquad (!)$$

- $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$
- $\sum_{k=1}^{n} k \cdot 2^k = 0 \cdot 1 + 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 8 + \cdots + n \cdot 2^n$
- $\int_a^b f(x)\, dx = \lim_{n\to\infty} \Delta x \cdot \sum_{i=1}^{n} f(x_i)$

$$A_1 \cup A_2 \cup \cdots \cup A_n = \textstyle\bigcup_{i=1}^{n} A_i \qquad \bigcup_{i=1}^{1} A_i = A_1 \qquad \bigcup_{i=1}^{0} A_i = \varnothing$$

$$A_1 \cap A_2 \cap \cdots \cap A_n = \textstyle\bigcap_{i=1}^{n} A_i \qquad \bigcap_{i=1}^{1} A_i = A_1 \qquad \bigcap_{i=1}^{0} A_i = U$$

triangular numbers $\boxed{\text{rij} \rightsquigarrow \text{reeks}}$



$$\begin{array}{cccccccc} 1 & +2 & +3 & +4 & +5 & +6 & \dots \\ 1 & 3 & 6 & 10 & 15 & 21 & \end{array}$$

$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$ $\quad S_n$

$$\begin{array}{c|cccccc} S_n & 1 & 2 & 3 & n-2 & n-1 & n \\ S_n & n & n-1 & n-2 & 3 & 2 & 1 \\ \hline 2S_n & n+1 & n+1 & n+1 & n+1 & n+1 & n+1 \end{array} \qquad 2S_n = n\cdot(n+1)$$

$$\sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$$

$$\begin{array}{cccccccc} 1 & +2 & +4 & +8 & +16 & +32 & +64 & \dots \\ 1 & 3 & 7 & 15 & 31 & 63 & 127 & \end{array}$$

$\sum_{k=0}^{n} 2^k = 1 + 2 + 4 + \cdots + 2^n \qquad S_n$

$$\begin{array}{ccccccc} 2S_n & & 2 & 4 & 2^{n-1} & 2^n & 2^{n+1} \\ S_n & 1 & 2 & 4 & 2^{n-1} & 2^n & \\ \hline S_n & -1 & & & & & 2^{n+1} \end{array} \qquad S_n = 2^{n+1} - 1$$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

rekenkundige reeks (*arithmetical*)

- $8, 11, 14, 17, 20, 23, \ldots$

$a_0 = a,\ a_1 = a + v,\ a_2 = a + 2v,\ \ldots$ verschil v

$$a_i = a + iv \quad (n \geqslant 0)$$

$$\boxed{\sum_{i=0}^{n} a_i = \frac{(n+1)\cdot(2a + n\,v)}{2}}$$

(aantal termen) x (eerste + laatste) / 2

- $8+11+14+17+\ldots+41 = \frac{12\cdot 49}{2} = 394 \qquad \frac{41-8}{11-8} = 11$

meetkundige reeks (*geometric*)

- $2, 6, 18, 54, 162, \ldots$

$a_0 = a,\ a_1 = ra,\ a_2 = r^2 a,\ \ldots$ reden r (*ratio*)

$$a_i = a\, r^i \quad (n \geqslant 0)$$

$$\boxed{\sum_{i=0}^{n} a\, r^i = a\,\frac{r^{n+1}-1}{r-1} \quad (r \neq 1)}$$

- $4+8+16+32+64 = 4\frac{2^5-1}{2-1} = 4\cdot 31 \qquad (r=2,\ a=a,\ n{+}1=5)$
- $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = \frac{1}{2}\frac{1-\frac{1}{2^4}}{1-\frac{1}{2}} = \frac{2^4-1}{2^4} = \frac{15}{16} \qquad (r=\frac{1}{2},\ a=\frac{1}{2},\ n{+}1=4)$
- $\underbrace{1-1}+\underbrace{1-1}+\cdots+1 = 1\frac{(-1)^{n+1}-1}{-1-1} \overset{n \text{ even}}{=} \frac{-2}{-2} = 1$

- $1+4+9+16+\cdots+n^2 = \sum_{i=1}^{n} i^2 = \dfrac{n\,(n+1)(2n+1)}{6}$

- $$\begin{array}{c|cccccccc} n & 1 & \times 2 & \times 3 & \times 4 & \times 5 & \times 6 & \times 7 & \dots \\ n! & 1 & 2 & 6 & 24 & 120 & 720 & 5040 & \end{array}$$

  $n! = 1\cdot 2\cdot 3\cdot \ldots \cdot n$ $0! = 1$ faculteit *factorial*

  $$n! = \begin{cases} 1 & n = 0 \\ n\cdot(n-1)! & n \geqslant 1 \end{cases}$$

- $$\begin{array}{c|ccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \\ F_n & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & \end{array}$$ Fibonacci

  $$F_n = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ F_{n-1} + F_{n-2} & n \geqslant 2 \end{cases}$$

Sch 1.8 Mathematical induction (later behandeld!)

END.