Datastructuren

Data Structures

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najaar 2024

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Set vs Dictionary

	{ key }	{ (key, value) }
	Set	Dictionary / Map
isEmpty	$A = \varnothing?$	
size	$\lvert A \rvert$	
isElement	$a \in A?$	$a \mapsto v(a)$
insert	$A \cup \{a\}$	(a, v)
delete	$A \setminus \{a\}$	a

ADT Set

- INITIALIZE: construct an empty set, return $A = \varnothing$.
- ISEMPTY: check whether there the set is empty ($A \stackrel{?}{=} \varnothing$, contains no elements).
- SIZE: return the number of elements, the cardinality $|A|$.
- ISELEMENT(a): returns whether a given object from the domain belongs to the set, $a \stackrel{?}{\in} A$.
- INSERT(a): add an element to the set, return $A \cup \{a\}$ (assume if it is not present?)
- DELETE(a): removes an element from the set, return $A \setminus \{a\}$ (assume it is present?).

ADT Dictionary / Map / Associative array

stores (*key*, *value*) pairs retrieval based on *key*

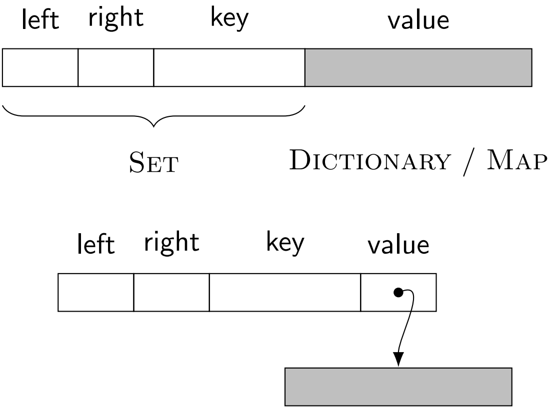
- INITIALIZE: construct an empty Map
- ISEMPTY: check whether there the Map is empty
- SIZE: return the number of elements
- RETRIEVE(*key*): returns *value* for (*key*, *value*) in Map (and signals when no such element present)
- INSERT(*key*, *value*): inserts the pair (what if another value for *key* is present?)
- DELETE(*key*): removes element with given *key*

list representations (of Set)

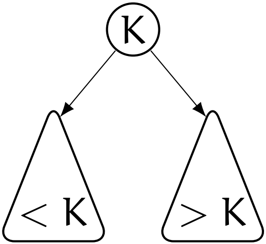
12	3	6	1	19	16	11	(unsorted) (array/linked)
1	3	6	11	12	16	19	(sorted)

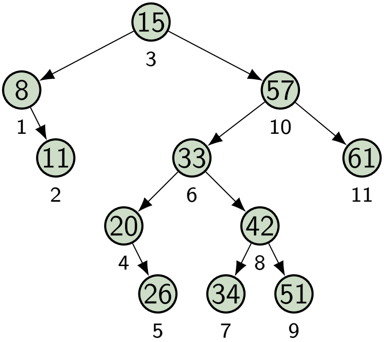
		$x \in S$	$S \cup \{x\}$	$S \setminus \{x\}$
array	unsorted	$\mathcal{O}(n)$	$\mathcal{O}(n) + \mathcal{O}(1)$	$\mathcal{O}(n) + \mathcal{O}(1)$
	sorted	$\mathcal{O}(\lg n)$	$\mathcal{O}(\lg n) + \mathcal{O}(n)$	$\mathcal{O}(n)$
linked	unsorted	$\mathcal{O}(n)$	$\mathcal{O}(n) + \mathcal{O}(1)$	$\mathcal{O}(n) + \mathcal{O}(1)$
	sorted	$\mathcal{O}(n)$	$\mathcal{O}(n) + \mathcal{O}(1)$	$\mathcal{O}(n) + \mathcal{O}(1)$
			locate+adapt	

binary tree Node



binary search tree



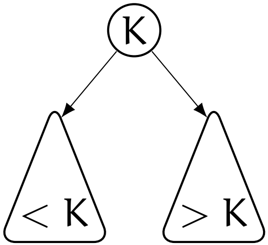


inorder: 8 11 15 29 26 33 34 42 51 57 61

BST and inorder

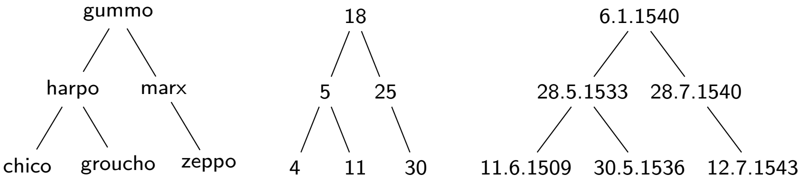
Lemma

Let T *be a binary search tree. Then the inorder traversal of* T *visits all the nodes in increasing order.*

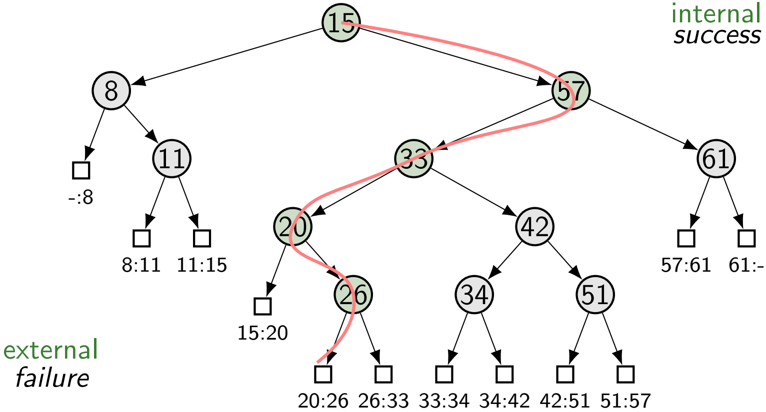


comparables

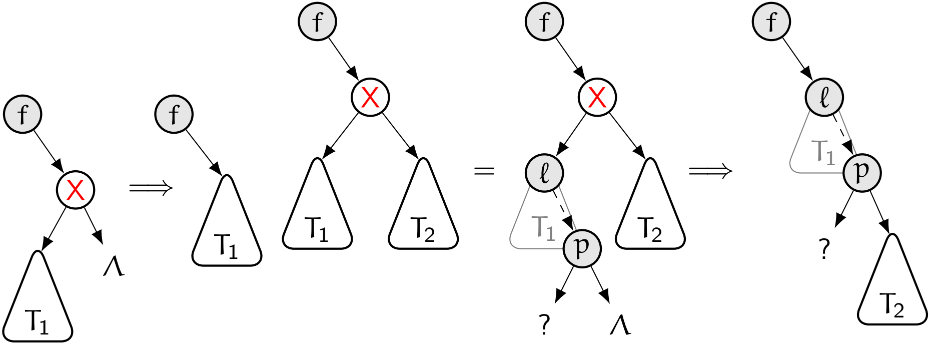
linear order on domain (keys)   gummo $<$ zeppo $18 < 30$



find, insert


internal
success
15
8
57
11
33
61
-:8
8:11
11:15
20
42
57:61
61:-
15:20
26
34
51
external
failure
20:26
26:33
33:34
34:42
42:51
51:57

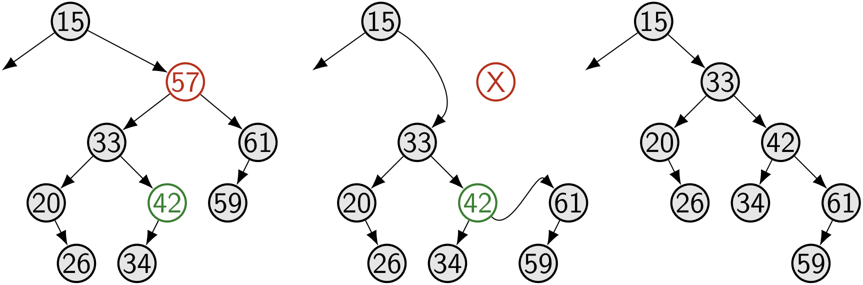
deletion 'by merging'



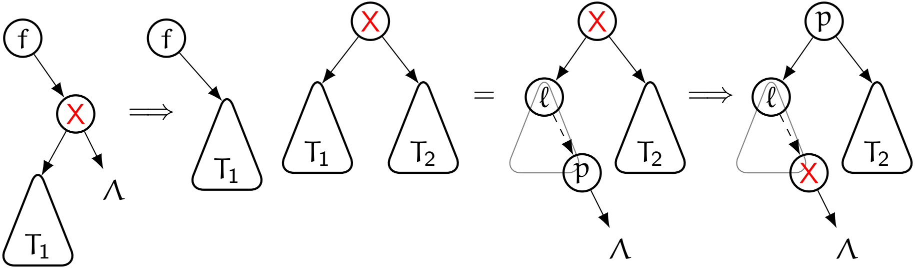
simple case: no [right] child otherwise link right child under predecessor

example: deletion by merging

deleting 57 predecessor 42



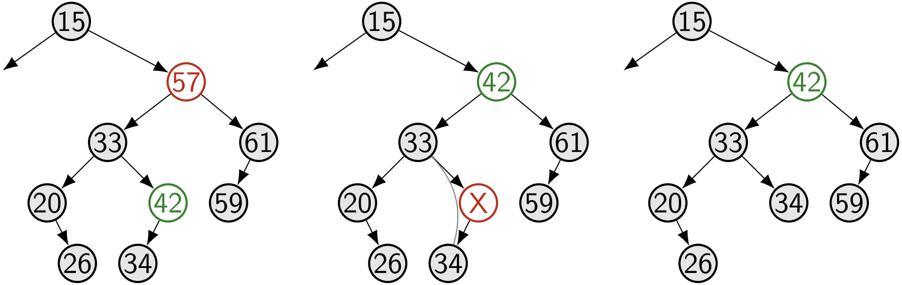
deletion ‘by copying’



simple case: no [right] child otherwise copy predecessor

example: deletion by copying

deleting 57 predecessor 42



Contents

search value/key

```
bool contains( const Comparable & x, Node *t ) const {
    if( t == nullptr )
        return false;
    else if( x < t->element )
        return contains( x, t->left );
    else if( t->element < x )
        return contains( x, t->right );
    else
        return true; // found
}
```

call with: `contains(v,root);`

find min/max value

```
BinaryNode * findMin( BinaryNode *t ) const {
    if( t == nullptr )
        return nullptr;
    if( t->left == nullptr )
        return t;
    return findMin( t->left );
}
BinaryNode * findMax( BinaryNode *t ) const {
    if( t != nullptr )
        while( t->right != nullptr )
            t = t->right;
    return t;
}
```

call with: `findMin(root);` and `findMax(root);`

insertion (recursive)

```
template<class T>
void Node<T>::insert(const T& el, Node<T> * & p) {
    if( p == nullptr ) {
        p = new Node{el, nullptr, nullptr};
    } else if (el < p->data) {
        insert(el, p->left);
    } else if (el > p->data) {
        insert(el, p->right);
    } else {
        ; // Duplicate; do nothing
    }
}
```

call with: `insert(el,root);`

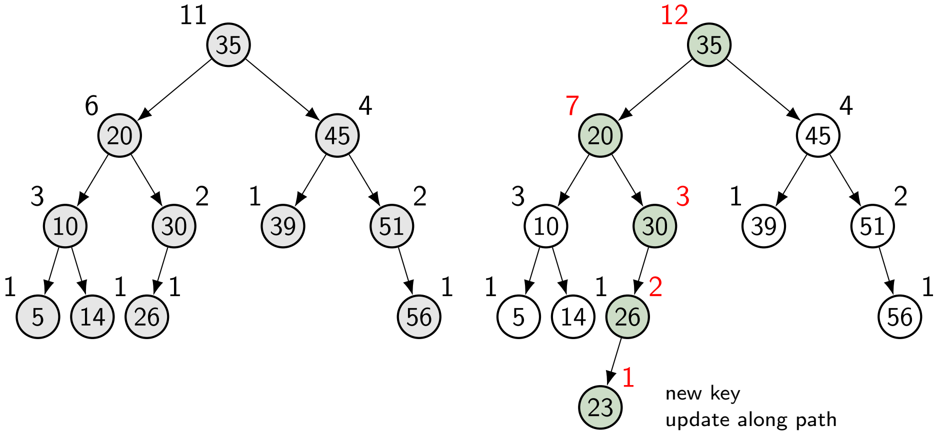
deletion (recursive)

```
void remove( const Comparable & x, Node * & t ) {
    if( t == nullptr )      return;
    if( x < t->data )       remove( x, t->left );
    else if( x > t->data)   remove( x, t->right );
    else if( t->left != nullptr && t->right != nullptr ) {
        Node *pred = findMax( t->left );
        t->element = pred->element;   \\ copying
        remove( t->element, t->left );
    }
    else {       \\ no two children
        BinaryNode *oldNode = t;
        if(t->left != nullptr ) t = t->left
        else                    t = t->right;
        delete oldNode;
    }
}
```

aanroepen met: `remove(el,root);`

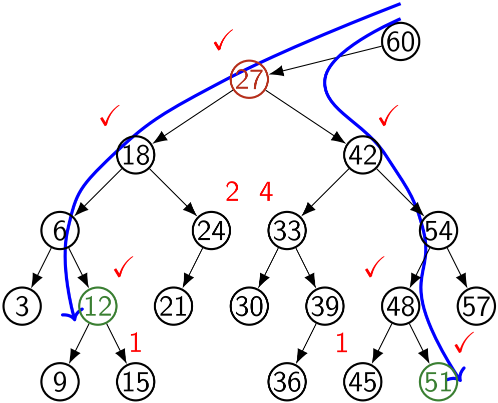
Contents

order statistics: find k-th element



augment each node with the size of its subtree

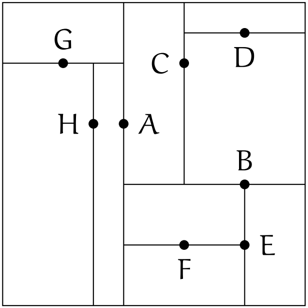
range search [12,52]

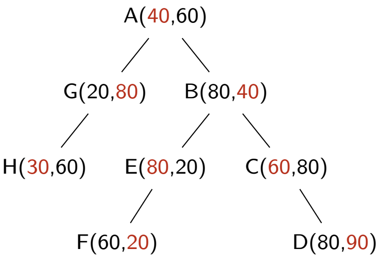


start counting where paths diverge

count trees 'inside' + nodes along path

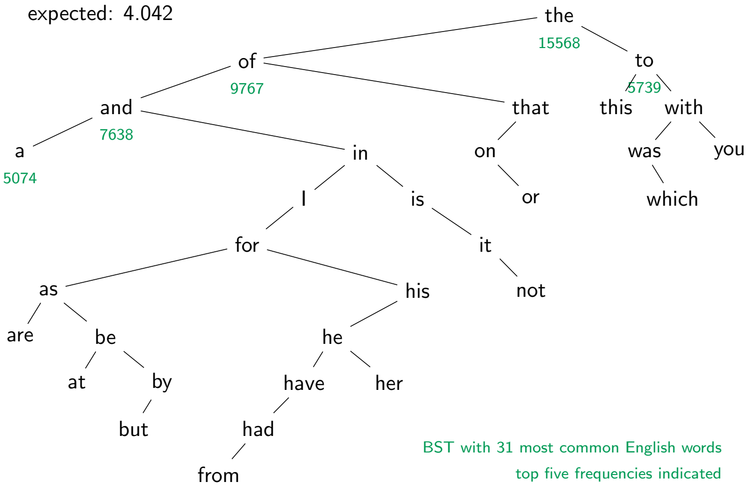
k-d-tree ⊠





Contents

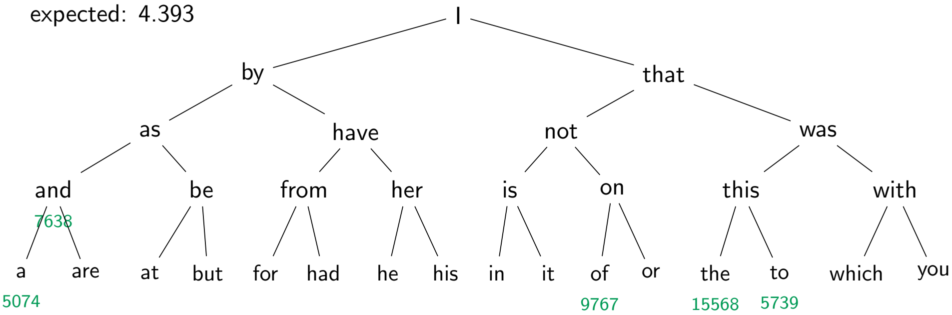
order of frequency



BST with 31 most common English words

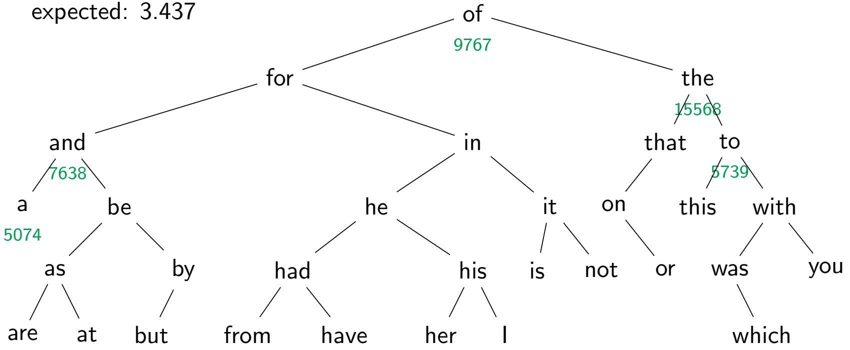
top five frequencies indicated

perfectly balanced



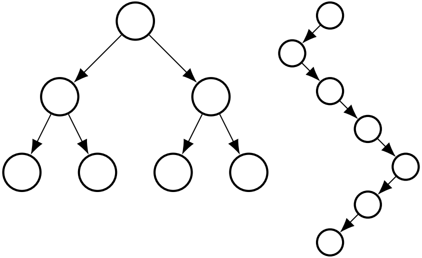
optimal

expected: 3.437



source: [Knuth] TAoCP Vol.3 (Sorting and Searching)

binary search tree BST



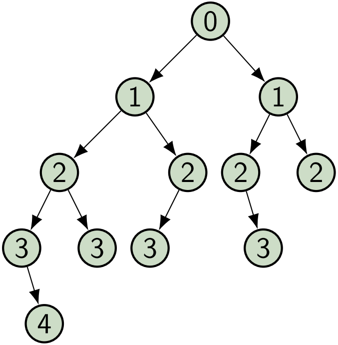
unsuccessful search in: (worst case search complexity)

- linear tree $O(n)$
- optimal tree $O(\lg(n))$ (complete tree)

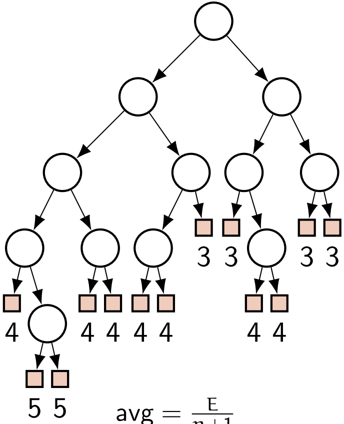
proof, and average case behaviour: see later

internal / external path length

successful search



failure



$\text{avg} = \frac{I}{n} + 1 = \frac{I+n}{n}$

$I = 0 + 2*1 + 4*2 + 4*3 + 1*4 = 26$

$\text{avg} = \frac{E}{n+1}$

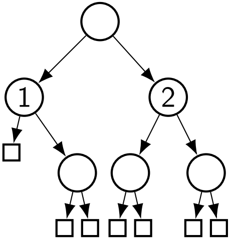
$E = 4*3 + 7*4 + 2*5 = 50$

relation I and E

Lemma

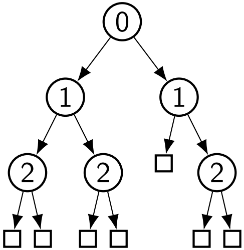
Let T be a binary tree with n nodes, with internal path length I and external path length E. Then $E = I + 2n$.

induction:



$$n = n_1 + n_2 + 1$$
$$I = I_1 + I_2 + n_1 + n_2$$
$$E = E_1 + E_2 + (n_1{+}1) + (n_2{+}1)$$
$$= (I_1 + 2n_1) + (I_2 + 2n_2) + n_1 + n_2 + 2$$
$$= \underbrace{(I_1 + I_2 + n_1 + n_2)}_{I} + \underbrace{(2n_1 + 2n_2 + 2)}_{2n}$$

internal/external path length



path length: count edges

n (internal) nodes success

ipl $I = 0 + 1 + 1 + 2 + 2 + 2 = 8$

$n + 1$ (external) leaves failure

epl $E = 3 + 3 + 3 + 3 + 2 + 3 + 3 = 20$

Definition (internal/external path length)

ipl $I =$ sum of path lengths to all nodes

epl $E =$ sum of path lengths to the 'extended' leaves

avg # comparisons *success*: $\frac{I}{n} + 1$ *failure*: $\frac{E}{n+1}$

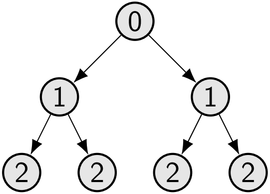
relation ipl/epl: $E = I + 2n$

extremal trees average per key

balanced (optimal)

h levels: $n = 2^h - 1$ nodes

$h = \lg(n+1)$

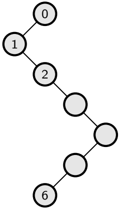


$I_n = \sum_{i=0}^{h-1} i \cdot 2^i$, $E_n = 2^h \cdot h$

$\Rightarrow I_n = (n+1)\lg(n+1) - 2n$

$\text{avg} = \frac{n+1}{n}\lg(n+1) - 1 \sim \lg n$

linear (worst case)

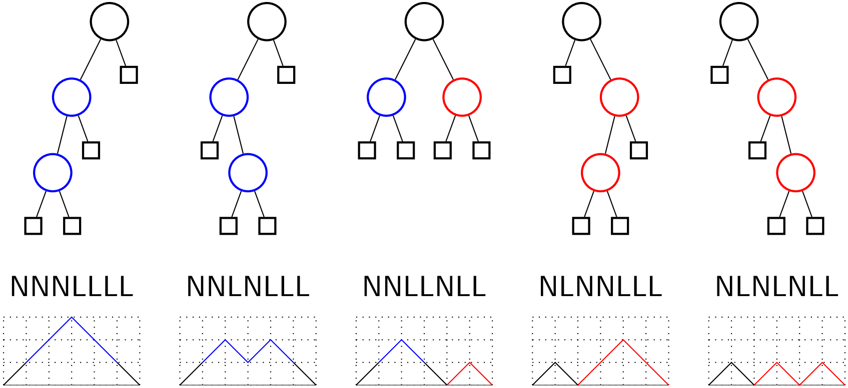


$I_n = \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$

$E_n = I_n + 2n = \frac{n(n+3)}{2}$

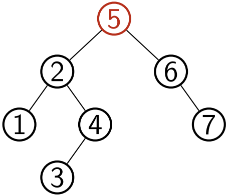
$\text{avg} = \frac{n+1}{2}$

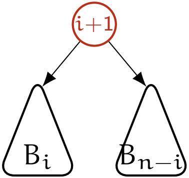
counting trees ☒



counting trees

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, . . .





1	1	2	5	14
14	5	2	1	1

(unlabeled) n-node binary trees

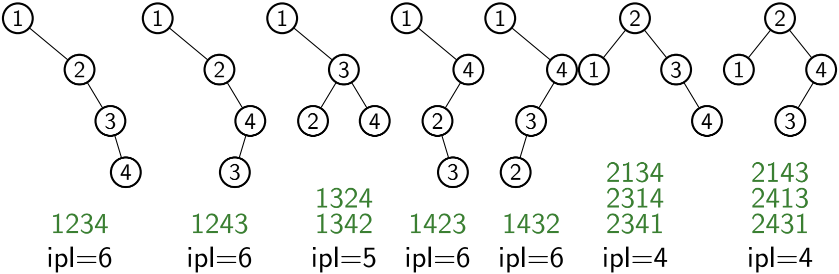
$B_{n+1} = \sum_{i=0}^{n} (B_i \cdot B_{n-i})$ with $B_0 = 1$

$B_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$ Catalan numbers

also the number of BST with given values

trees and permutations

'*average tree*' adding (permutation) keys $\Rightarrow$ tree structure



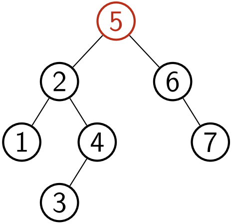
$4! = 24$ permutations but $\mathrm{B}_4 = 14$ BST's

average ipl: $\frac{1}{24}(12 \times 4 + 4 \times 5 + 8 \times 6) = \frac{116}{24} = \frac{29}{6}$

average ipl BST ⊠

I_n *average internal path length* over all BST with n nodes

average over permutations
permutation $1, \ldots, n$ into BST $\Rightarrow$ tree structure



permutation
determines left & right subtrees

5   2 4 1 3 $\sim I_4$
     6 7 $\sim I_2$

any k root = first element

$$I_n = (n-1) + \frac{1}{n} \sum_{k=1}^{n} (I_{k-1} + I_{n-k})$$

telescope! ☒

I_n average internal path length n nodes

so	$I_n = (n-1) + 2(I_0 + I_1 + \cdots + I_{n-1})/n$
also	$I_{n-1} = (n-2) + 2(I_0 + I_1 + \cdots + I_{n-2})/(n-1)$
subtract	$n\, I_n - (n-1) I_{n-1} = 2n - 2 + 2I_{n-1}$
thus	$n\, I_n = (n+1) I_{n-1} + 2n - 2$

$$\frac{I_n}{n+1} = \frac{I_{n-1}}{n} + \frac{2}{n+1} - \frac{2}{n(n+1)}$$

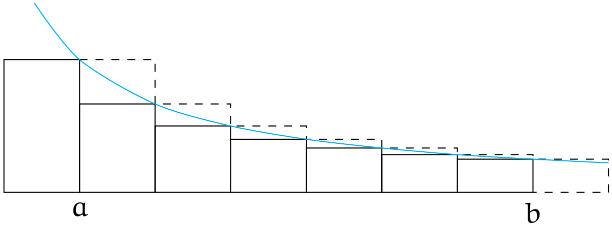
$$\frac{I_{n-1}}{n} = \frac{I_{n-2}}{n-1} + \frac{2}{n} - \frac{2}{(n-1)n}$$

$$\cdots$$

$$\frac{I_1}{2} = \frac{I_0}{1} + \frac{2}{2} - \frac{2}{1 \cdot 2}$$

$$\frac{I_n}{n+1} = \frac{I_0}{1} + O(\ln n) - \frac{2n}{n+1}$$

afschatten



$$\int_{a}^{b-1} f(x)\,dx \leqslant \sum_{k=a}^{b} f(k) \leqslant \int_{a-1}^{b} f(x)\,dx$$

$$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = 1 + \sum_{k=2}^{n} \frac{1}{k} \leqslant 1 + \int_{1}^{n} \frac{1}{x}\,dx = 1 + \lg n$$

end.