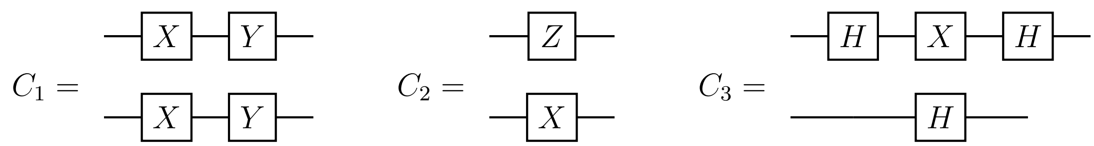
Applied Quantum Algorithms assignment 2: circuits and quantum algorithms — Due April 3rd, 2020

## Outline

This assignment consists of 9 questions, each carrying 1 or 2 points, 16 points in total. It constitutes 50% of your total assignment mark (25% of the overall mark for this course).

**The deadline the assignment is the beginning of the lecture 20th of March 2020.**

1. (2 points) Consider the following two circuits:




   (a) Express $C_1$, $C_2$ and $C_3$ using standard linear-algebraic notation (i.e., using $X, Y, Z, H$ to denote the operators, tensor products $\otimes$ and standard matrix products). (b) Simplify the expression $C_1 - C_2 + \sqrt{2}C_3$ using the rules of tensor algebra, and the relations between Pauli operators and the Hadamard operator.

2. (2 points) (a) Draw the circuit diagram for the quantum Fourier transform (QFT) over $n$ qubits. You can use the gate set which includes the controlled-Z rotations by any angle. Don't forget swaps! (b) what is the gate-complexity of QFT with respect to the gate set which includes the Hadamard, swap and controlled-Z rotations. (c) what is the depth complexity of the QFT algorithm (relative to the drawing you provided)? State the complexities as functions of $n$.

3. (1 point) Define the input and output of the ($t$-ancilla) quantum phase estimation algorithm.

4. (2 points) (a) Draw the circuit for quantum phase estimation (QPE) for the $n$-qubit unitary $U$ with $\ell$-ancilla qubits (in the eigenvalue-carrying register), assuming access to the unitary ctrl-$U$. You can draw the quantum Fourier transform and ctrl-$U^k$'s as single gates. (b) What is the gate-complexity of the circuit for the quantum phase estimation algorithm, including the cost of QFT? (c) What is the depth-complexity of this circuit? In both questions (a) and (b) make explicit which part comes from the QFT. Also, state both complexities as functions of $\ell$ and $n$. You can assume that the gate and depth complexities of ctrl-$U$ are some functions $f_g(n)$ and $f_d(n)$, and you can use that the costs of powers of ctrl-$U$ will be multiples of these functions. For

simplicity, in this question you can consider ctrl-$U$ a part of the gate set, i.e., you can assume that $f_g(n) = f_d(n) = 1$.

5. (2 points) (a) Draw the circuit for one round of the single-qubit quantum phase estimation (QPE) for the time evolution $U = e^{iHt}$ in which you apply the power $U^k$ to the eigenvector-carying register controlled on the ancilla qubit, assuming access to the controlled time evolution $U$ as a black box. (b) What is the gate-complexity of the circuit used in this round of the single-qubit quantum phase estimation algorithm? (c) What is the depth complexity of this circuit? You can assume that the gate and depth complexities of the controlled time evolution $U = e^{iHt}$ are some functions $f_g(n,t)$ and $f_d(n,t)$ (note that the cost will depend on the evolution time $t$) and you can use that the gate and depth complexity of the controlled $U^k$ are given by $f_g(n,kt)$ and $f_d(n,kt)$, respectively. (d) For extra credit, express the total complexities more precisely by filling in the complexities of a chosen Hamiltonian simulation algorithm for $f_g(n,t)$ and $f_d(n,t)$.

6. (2 points) Give the expression for the probability of measuring 0 on the ancilla qubit for the round in the single ancilla QPE as discussed in question 5, assuming the system (i.e., eigenvector-carying) register is prepared in the state $\sum_j a_j|\lambda_j\rangle$ (where the pairs $\{(\lambda_j, |\lambda_j\rangle)\}$ are the eigenvalue-eigenvector pairs of the Hamiltonian $H$)?

7. (1 point) Draw a circuit to apply $e^{i\frac{\pi}{4}X_0\otimes X_1}$ to a two-qubit system.

8. (2 points) (a) Write the first-order Suzuki-Trotter approximation for

$$e^{ih_0(X_0X_1+X_1X_2)+ih_1(Z_0+Z_1+Z_2)}, \tag{1}$$

and (b) calculate the error in the case of the first-order Trotter approximation.

9. (2 points) Draw the circuit which implements the Trotterized unitary from Question 8.