NEXT TWO LECTURES:

QUANTUM COMPUTING BOOTCAMP

→ BASICS YOU WILL NEED LATER ON

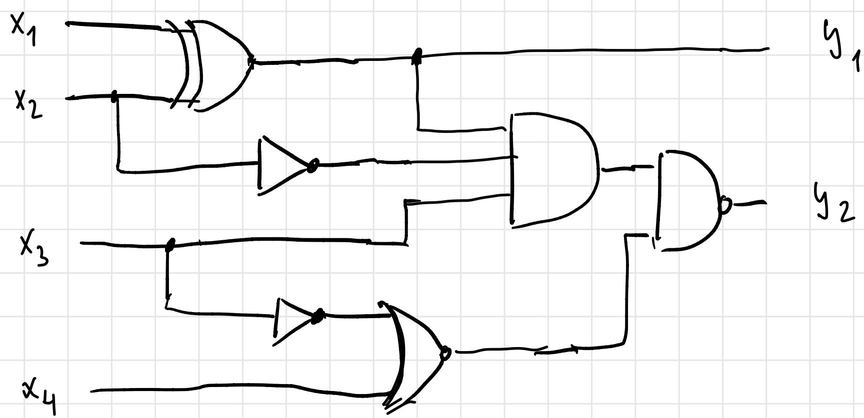
→ CERTAIN ADVANCED TOPICS FOR CONTEXT

... IT WILL BE QUITE INTENSIVE ...

BUT NOT ALL IS NECESARY FOR SECOND HALF OF COURSE

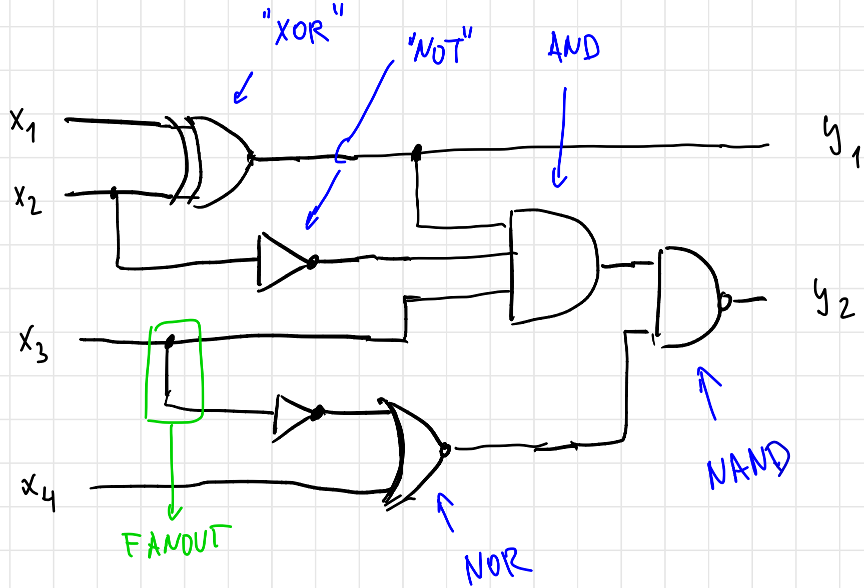
A DIGITAL COMPUTATION

- REDUCES TO MANIPULATION OF BITS USING SIMPLE OPERATIONS



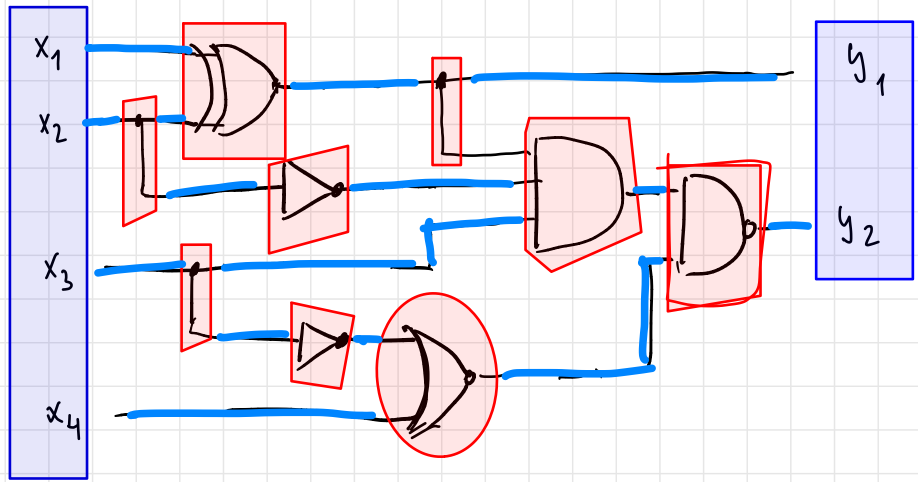
A DIGITAL COMPUTATION

- REDUCES TO MANIPULATION OF BITS USING SIMPLE OPERATIONS



- BOOLEAN CIRCUIT
- $f : \{0,1\}^n \longrightarrow \{0,1\}^m$
- $f(\vec{x}) = \left(XOR(x_1,x_2) \,,\, NAND\left(AND\left(XOR(x_1,x_2), \bar{x}_2, x_3 \right), NOR\left(\bar{x}_3, x_4 \right) \right) \right)$

A DIGITAL COMPUTATION



- **ELEMENTS**:
  - INPUT (MEMORY) REGISTER
  - GATES (GATE SET)
  - OUTPUT REGISTER

IMPORTANT PROPERTIES

- UNIVERSALITY OF GATE SET $\mathcal{G}$ (A.K.A FUNCTIONAL COMPLETENESS)

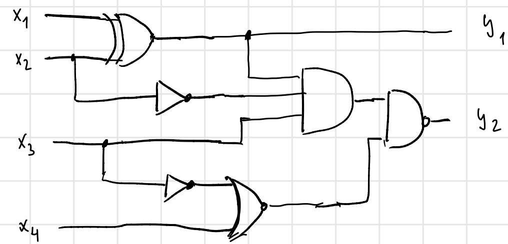
  $\equiv$ SUFFICES TO IMPLEMENT ANY $f: \{0,1\}^n \to \{0,1\}^m$

  HOLDS E.G. FOR $\mathcal{G} = \{$ FANOUT, AND, NOT, XOR $\}$, $\mathcal{G} = \{$ FANOUT, NAND $\}$

- ALL BINARY MAPPINGS SUFFICE... BITSTRINGS → INTEGERS →, FLOAT/DOUBLE →, STRINGS → ANY COMPUTATION

DIFFERENT REPRESENTATIONS OF COMPUTATIONS

CIRCUITS:



(BOOLEAN) ALGEBRA: $\vec{y} = f(\vec{x}) = \left(XOR(x_1, x_2) \, , \, NAND\left(AND(XOR(x_1,x_2), \bar{x}_2, x_3), NOR(\bar{x}_3, x_4) \right) \right)$

PROGRAM IN LANGUAGE:
("PSEUDO ASSEMBLY")

CCA...

```
XOR    X1  X2  X3
CP     X2  X4
CP     X5  X6
NEG    X4
NEG    X6
MAND   3   X1  X4  X5  X2
OR     X6  X7  X6
NEG    X6
AND    X2  X6  X6
NEG    X6
RETURN X2  X6
```

CIRCUITS $\longrightarrow$ ALGORITHMS

- CIRCUITS := SPECIFICATION OF ONE COMPUTATION $n \to m$ (FIXED)
  := LANGUAGE DESCRIBING IN$\to$OUT BEHAVIOR

- ALGORITHM

  TYPICALLY HAS A PURPOSE . . .

  - (COMPUTATIONAL) PROBLEM /TASK $\mathcal{T}_j$ HAS INSTANCE SIZE OF $\mathcal{T}$

    PROBLEM /TASK $\downarrow$ E.G. "SUDOKU PUZZLE"

    INSTANCE $\downarrow$ E.G. ONE PARTICULAR SUDOKU PUZZLE

    SIZE $\searrow$ DIMENSION OF THIS PARTICULAR PUZZLE (E.G. 9×9)

CIRCUITS → ALGORITHMS

- CIRCUITS := SPECIFICATION OF ONE COMPUTATION $n \to m$ (FIXED)
  := LANGUAGE DESCRIBING IN→OUT BEHAVIOR

- ALGORITHM
  - (COMPUTATIONAL) PROBLEM / TASK $\mathcal{T}$; HAS INSTANCE SIZE $\underline{n}$ (SIZE OF $\mathcal{T}$)
  - ALGORITHM SOLVING $\mathcal{T}$ IS A (COMPUTABLE) "RECIPE" WHICH TAKES n ON INPUT AND OUTPUTS CIRCUIT $\mathcal{C}_n$ SOLVING INSTANES OF SIZE n
  - "RECIPE" ≈ ANOTHER CIRCUIT / TURING MACHINE

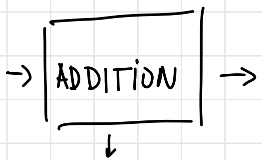
CIRCUITS → ALGORITHMS

- CIRCUITS := SPECIFICATION OF ONE COMPUTATION $n \to m$ (FIXED)
  := LANGUAGE DESCRIBING IN→OUT BEHAVIOR

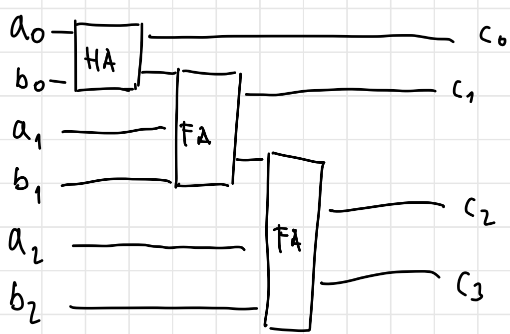
- ALGORITHM

E.G. $\mathcal{T}$ = BINARY ADDITION ; SIZE : n - BITS

SIZE 3



$$
\begin{array}{r}
(a_2\, a_1\, a_0) \\
+ \quad (b_2\, b_1\, b_0) \\
\hline
(c_3\, c_2\, c_1\, c_0)
\end{array}
$$

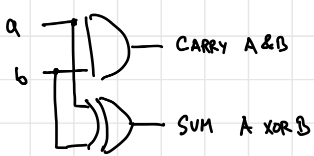


"PATTERN" OBVIOUS . .

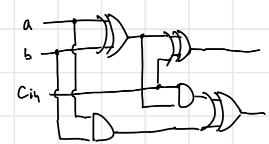
"SIMPLE" ALGORITHM

"PATTERN" = ALGORITHM . . .

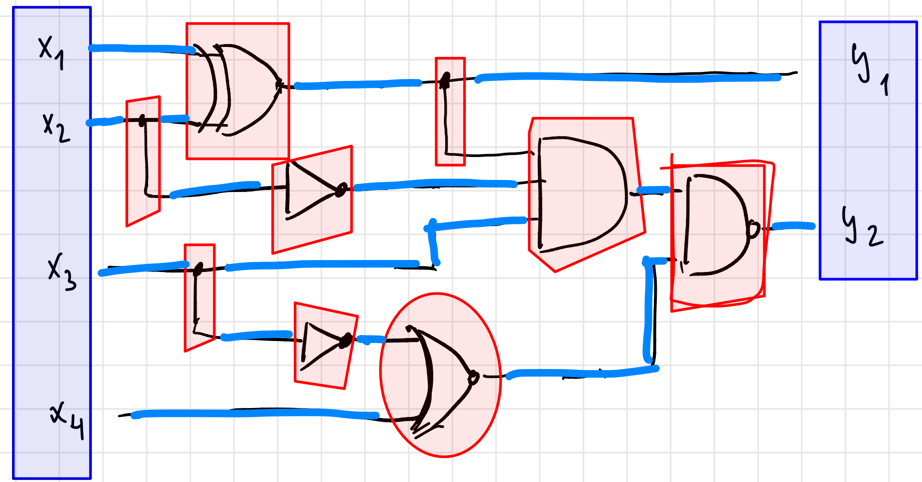
$\boxed{HA}$:=



$\boxed{FA}$:=



FROM LOGIC (ABSTRACTION) TO PHYSICS...



- PHYSICAL REALIZATION OF REGISTERS := COLLECTIONS OF <u>BITS</u>

  - ANY TWO LEVEL SYSTEM (OR $n \geq 2$)
  - OPTICAL (LIGHT/DARK; POLARIZATION; FREQUENCY?)
  - ELECTRICAL (VOLTAGE / CURRENT)
  - MECHANICAL ...

BUT ALSO:

- WAYS TO MANIPULATE BIT STATES (CONDITIONALLY)

FLIP BIT 2 IFF $BIT_1 = 1$.

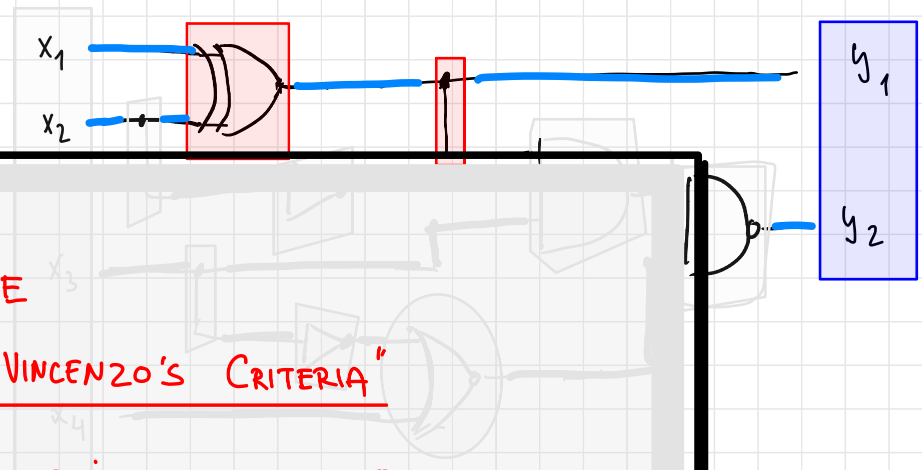
- READOUT ("RETURN y_1 y_2")
  → "MEASUREMENT"

- "LOADING" (SET $X_1 .. X_n$ to ANY VALUE
  00..0 WILL SUFFICE)
  → "INITIALIZATION"

FROM LOGIC (ABSTRACTION) TO PHYSICS...

- PHYSICAL REALIZATION OF REGISTERS := COLLECTIONS OF BITS

  - ANY TWO LEVEL SYSTEM (or n≥2)
  - OPTICAL (LIGHT/DARK; POLARIZATION; FREQUEN...
  - ELECTRICAL (VOLTAGE / CURRENT)
  - MECHANICAL ...



SEE

"Di VINCENZO'S CRITERIA"

FOR BUILDING A Q.C.

BUT ALSO:

- READOUT ("RETURN y_1 y_2")

MEASUREMENT

- WAYS TO MANIPULATE BIT STATES (CONDITIONALLY)

FLIP BIT 2 IFF $\text{BIT}_1 = 1$.

- "LOADING" (SET $X_1 .. X_n$ to ANY VALUE 00..0 WILL SUFFICE)
  → "INITIALIZATION"

NEXT ... TREAT EACH ELEMENT AS A QUANTUM-PHYSICAL SYSTEM...

ELEMENTS		POSTULATES OF QUANTUM MECHANICS
• REGISTER OF BITSTRINGS	$\longrightarrow$	(1) "STATE SPACE"
• MANIPULATION	$\longrightarrow$	(2) EVOLUTION
• READOUT	$\longrightarrow$	(3) MEASUREMENT POSTULATE
• FROM BIT TO BITSTRINGS	$\longrightarrow$	(4) COMPOSITE SYSTEMS

BASED ON NIELSEN & CHUANG...

TWEAKS POSSIBLE ...

NEXT... TREAT EACH ELEMENT AS A QUANTUM-PHYSICAL SYSTEM...

ELEMENTS OF COMPUTING		QM POSTULATES		ELEMENTS OF LINEAR ALGEBRA
• REGISTER OF BITSTRINGS $\longrightarrow$	(1)	"STATE SPACE"	$\longleftrightarrow$	VECTOR SPACES
• MANIPULATION $\longrightarrow$	(2)	EVOLUTION	$\longleftrightarrow$	SPECIAL LIN. OPS. & FUNCTIONS OF OPS
• READOUT $\longrightarrow$	(3)	MEASUREMENT POSTULATE	$\longleftrightarrow$	INNER PRODUCTS
• FROM BIT TO BITSTRINGS $\longrightarrow$	(4)	COMPOSITE SYSTEMS	$\longleftrightarrow$	TENSOR PRODUCTS (KRONECKER PRODUCT)

PLAYS ROLE OF

"LOGIC" ; "ALGEBRA"

POSTULATE 1. STATE SPACE (REGISTER)

"THE STATE OF ANY CLOSED PHYSICAL SYSTEM IS COMPLETELY DESCRIBED BY A STATE VECTOR : A UNIT VECTOR IN A (COMPLEX) HILBERT SPACE $\mathcal{H}$"

HERE : $\mathcal{H} = \mathbb{C}^N$ (only finite dimensional ...)

IMPLEMENTATIONS : A SPIN OF ELECTRON, POLARIZATION OF LIGHT
VOLTAGE/CURRENT, ENERGY OF QUANTIZED H. OSCILLATOR

POSTULATE 1. STATE SPACE → THE MATH

$\mathcal{H} = \mathbb{C}^N$ IN A NUTSHELL

- $\alpha_k \in \mathbb{C}$ $\quad \alpha_k = a + bi \;;\; a, b \in \mathbb{R} \;,\; i = \sqrt{-1}$

- $\mathbb{C}^N \ni \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} = \alpha_0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + \alpha_{N-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

BASIS: $\vec{e}_0 \quad \vec{e}_1 \quad \vec{e}_{N-1}$ $\quad \Rightarrow \mathbb{C}^N \ni \vec{x} = \sum_{k=0}^{N-1} \alpha_k \vec{e}_k$

BASIS B ; $|B| = N$

DIRAC (BRA-KET): $|\psi\rangle \quad |0\rangle \quad |1\rangle \quad |N-1\rangle$ $\quad \Rightarrow |\psi\rangle = \sum_{k=0}^{N-1} \alpha_k |k\rangle$

NORM: $\| |\psi\rangle \|_2 = \sqrt{\sum_{k=0}^{N-1} |\alpha_k|^2}$ $\qquad$ UNIT $\Rightarrow \| |\psi\rangle \|_2 = 1$

POSTULATE 1. STATE SPACE

> "THE STATE OF ANY CLOSED PHYSICAL SYSTEM IS COMPLETELY DESCRIBED BY A STATE VECTOR: A UNIT VECTOR IN A (COMPLEX) HILBERT SPACE $\mathcal{H}$"

QC INTERPRETATION:

n-bit REGISTER $\longrightarrow$ 2^n configuration/states $\overset{00..00, .., 111..11}{\nearrow}$ $0, 1, 2 \dots, 2^n-1$ —— $|0\rangle, |1\rangle \dots |2^n-1\rangle \longrightarrow \sum_{k=0}^{2^n-1} \alpha_k |k\rangle$

CLASSICAL STATE $0, 1, 2 \dots N-1$

QUANTUM STATE $|0\rangle, |1\rangle, |2\rangle \dots |N-1\rangle$

"(QUANTUM) AMPLITUDE"

BUT ALSO $|\psi\rangle = \sum_{k=0}^{N-1} \alpha_k |k\rangle$ WITH $\sum_k |\alpha_k|^2 = 1$

POSTULATE 1. STATE SPACE (REGISTER)

"THE STATE OF ANY CLOSED PHYSICAL SYSTEM IS COMPLETELY DESCRIBED BY A STATE VECTOR: A UNIT VECTOR IN A (COMPLEX) HILBERT SPACE $\mathcal{H}$"

QC INTERPRETATION:

n-bit REGISTER $\longrightarrow$ 2^n configuration/states $0, 1, 2 \dots, 2^n-1$ (00..00, .., 111..11) — $|0\rangle, |1\rangle \dots |2^n-1\rangle \longrightarrow \sum_{k=0}^{2^n-1} \alpha_k |k\rangle$

CLASSICAL STATE $0, 1, 2 \dots N-1$

QUANTUM STATE $|0\rangle, |1\rangle, |2\rangle \dots |N-1\rangle$

"SUPERPOSITION"
(MATH: LINEAR COMBINATION)

BUT ALSO $|\psi\rangle = \sum_{k=0}^{N-1} \alpha_k |k\rangle$ WITH $\sum_k |\alpha_k|^2 = 1$

NORMALIZATION
(UNIT NORM)

POSTULATE 2 - EVOLUTION (MANIPULATING REGISTER)

VERSION (1) [DISCRETE TIME]

THE DISCRETE TIME EVOLUTION (FROM t_1 to t_2) OF A CLOSED SYSTEM IS DESCRIBED BY A UNITARY TRANSFORMATION (MAP, OPERATOR) U

$$|\psi_{t_2}\rangle = U |\psi_{t_1}\rangle$$

POSTULATE 2 EVOLUTION → THE MATH

$M : \mathbb{C}^N \to \mathbb{C}^N$;

$M\left(\sum \alpha_j |\psi_j\rangle\right) = \sum \alpha_j M|\psi_j\rangle$ (LINEAR) ; $M \in \mathcal{L}(\mathbb{C}^N)$

GIVEN A BASIS : $M \Leftrightarrow \begin{bmatrix} m_{0,0} & \cdots & m_{0,(N-1)} \\ \vdots & & \vdots \\ m_{(N-1),0} & & m_{(N-1),(N-1)} \end{bmatrix}$ $m_{ij} \in \mathbb{C}$

CONJUGATE TRANSPOSE : $M \mapsto M^{\dagger}$ (IN LIN. ALG. M^*)
"DAGGER"

$$\begin{bmatrix} m_{0,0} & \cdots & m_{0,N-1} \\ \vdots & & \vdots \\ m_{N-1,0} & & m_{(N-1,N-1)} \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{m_{00}} & \cdots & \overline{m_{N-1,0}} \\ \vdots & & \\ \overline{m_{0,N-1}} & \cdots & \overline{m_{N-1,N-1}} \end{bmatrix}$$

(TRANSPOSE & CONJUGATE)

UNITARY U IFF $UU^{\dagger} = U^{\dagger}U = \mathbb{1}$

$\mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

HERMITIAN H IFF $H = H^{\dagger}$

UNITARY : NORM PRESERVING

$\| |\psi\rangle \| = \| U|\psi\rangle \| \quad \forall\, U, |\psi\rangle$

POSTULATE 2 - EVOLUTION (GATES)

VERSION (1) [DISCRETE TIME]

THE DISCRETE TIME EVOLUTION (from t_1 to t_2) OF A CLOSED SYSTEM IS DESCRIBED BY A UNITARY TRANSFORMATION (MAP, OPERATOR) U

$$|\psi_{t_2}\rangle = U|\psi_{t_1}\rangle$$

NB: UNITARIES ARE ALL LINEAR MAPS MAPPING $\underbrace{\text{(PURE) QUANTUM STATES}}_{\text{norm-1 vector in } \mathbb{C}^N}$ TO (PURE) Q. STATES

POSTULATE 2 - EVOLUTION

$$\boxed{\begin{array}{l}\text{VERSION 2 (time-independent, dynamic Schrödinger Equation)} \\ \text{THE CONTINUOUS TIME EVOLUTION OF A CLOSED SYSTEM IS DESCRIBED} \\ \text{BY THE SCHRÖDINGER EQUATION} \\ (1) \quad i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \; ; \; H = \text{Hermitian HAMILTONIAN}\end{array}}$$

NB: BY SOLVING THE DIFF. EQ. (1) INTEGRATING FROM t_1 to t_2

WE GET

$$|\psi(t_2)\rangle = U(t_1, t_2)|\psi(t_1)\rangle \quad \text{WITH} \quad \boxed{\begin{array}{l} U(t_1,t_2) := U = \exp(-iH\Delta t) \\ \Delta t = t_2 - t_1 \end{array}}$$

"NATURAL UNITS" $\rightarrow$

$\hbar = 1 \ldots$ (USE IN HW)

MATH LOOKUP:

- FUNCTIONS OF OPERATORS : $M, M^2, \ldots M^k$
- $f(x) = \sum a_k x^k$

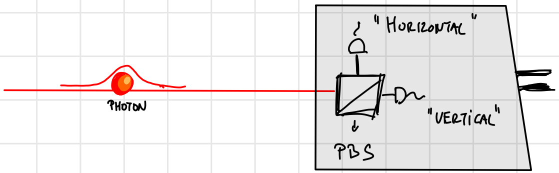
$\Rightarrow f(M) = \sum \alpha_k M^k$

$$\exp(x) = \sum_k \frac{x^k}{k!} \; ; \; \Rightarrow \exp(M) = \sum_k \frac{M^k}{k!}$$

CAN CHECK : $\exp(iM)$ is UNITARY IFF M is HERMITIAN...

→ USE SPECTRAL THEOREM... (TUTORIALS + TAKE HOME ASSIGNMENT)

POSTULATE 3 : MEASUREMENT (READOUT)



SCREEN SAYS "H" OR "V"

VERSION (1) : PROJECTIVE ORTHONORMAL BASIS MEASUREMENTS ASSOCIATED TO AN OBSERVABLE (NON-DEGENERATE)

LET $\mathcal{H} = \mathbb{C}^N$ BE THE STATE SPACE OF A Q.M. SYSTEM S

LET $\mathcal{M} = \{ |\varphi_i\rangle \}_{i=0}^{N-1}$ BE AN INDEXED ORTHONORMAL BASIS. $(i \in I = \{0,1 \dots N-1\})$

ANY $\mathcal{M}$ SPECIFIES AN ALLOWED Q. MEASUREMENT WHICH WHEN APPLIED TO $|\psi\rangle$ RETURNS OUTCOME $i \in I$ WITH PROBABILITY $\boxed{P_{|\psi\rangle}(i) = |\langle \varphi_i | \psi \rangle|^2}$

BORN RULE

ORTHONORMAL BASIS := O.N.B.

MATH AROUND MEASUREMENTS

- $(|\psi\rangle, |\varphi\rangle) \longmapsto \langle\psi|\varphi\rangle$ (INNER PRODUCT)

  $|\psi\rangle = \sum \alpha_i |i\rangle$, $|\varphi\rangle = \sum \beta_i |i\rangle$, $\langle\psi|\varphi\rangle = \sum_i \overline{\alpha_i}\, \beta_i$

  CONJUGATE! SESQUILINEAR!

  NB: $|\psi\rangle \perp |\varphi\rangle \iff \langle\psi|\varphi\rangle = 0$

  $\| |\psi\rangle \| = \sqrt{\langle\psi|\psi\rangle}$

- SPECTRAL THEOREM

  VER 1: IF M is UNITARY OR HERMITIAN, $\exists$ O.N.B. $\{|\varphi_i\rangle\}_i$

  S.T. $M = \sum \lambda_i |\varphi_i\rangle\langle\varphi_i|$

  WHERE $|\varphi_i\rangle\langle\varphi_i| = \mathrm{Proj}^{|\varphi_i\rangle}$ def. with

  $|\varphi_i\rangle\langle\varphi_i| \,.\, |\psi\rangle := \langle\varphi_i|\psi\rangle\, |\varphi_i\rangle$

  (NB $|\varphi_i\rangle\langle\varphi_i|$ = "outer product" $= \vec{x}\,.\,\vec{x}^T = \begin{bmatrix} \vdots \end{bmatrix} \begin{bmatrix} \cdots \end{bmatrix} = \begin{bmatrix} \vdots & \cdots \end{bmatrix}$)

MATH AROUND MEASUREMENTS

- SPECTRAL THEOREM

VER 1: IF M is UNITARY OR HERMITIAN, $\exists$ O.N.B. $\{|\varphi_i\rangle\}_i$

S.T. $M = \sum \lambda_i |\varphi_i\rangle\langle\varphi_i|$

WHERE $|\varphi_i\rangle\langle\varphi_i| = \mathrm{Proj}^{|\varphi_i\rangle}$ · def. with

$|\varphi_i\rangle\langle\varphi_i| \, . \, |\psi\rangle := \langle\varphi_i|\psi\rangle \, |\varphi_i\rangle$

(NB $|\varphi_i\rangle\langle\varphi_i|$ = "outer product" $= \vec{x} \, . \, \vec{x}^T = [\vdots][\dots] = [\vdots \; \ddots]$)

$\lambda_i \equiv$ eigenvalues ; $|\varphi_i\rangle \equiv$ eigenvectors

NB: BASIS CHANGE is UNITARY... $\mathcal{M}_1 = \{|\varphi_i\rangle\}_i$; $\mathcal{M}_2 \{|\psi_i\rangle\}_i$ ONBs

$= \exists : U$ s.t. $\forall i \quad |\psi_i\rangle = U |\varphi_i\rangle$

VER 2: IF M is Hermitian/UNITARY, $\exists$ U st.

$$M = U \Delta U^\dagger \qquad \Delta = \mathrm{diag}(\lambda_1, \lambda_2 \dots \lambda_N)$$

" M diagonalizes in some ONB".

"CANONICAL" (COMPUTATIONAL) BASIS MEASUREMENT $\mathcal{M}_1 = \{|0\rangle, |1\rangle .. |N-1\rangle\}$

$$|\psi\rangle = \sum_k \alpha_k |k\rangle \implies P_{\mathcal{M}_1}(k) = |\langle k|\psi\rangle|^2 = |\alpha_k|^2$$

OTHER BASIS? $\mathcal{M}_2 = \{|\varphi_i\rangle\}_i \Rightarrow \exists\ \underset{\substack{\uparrow \\ \mathcal{M}_2}}{U|\varphi_i\rangle} = \underset{\substack{\uparrow \\ \mathcal{M}_1}}{|i\rangle}$

$$|\psi'\rangle = U|\psi\rangle = \sum_k \gamma_k |k\rangle \Rightarrow P_{\mathcal{M}_2}(k) = |\langle\varphi_i|\psi\rangle|^2 \stackrel{?}{=} |\underbrace{\langle k|U}_{\langle k|U = (U^\dagger|k\rangle)^\dagger}|\psi\rangle|^2 \stackrel{?}{=} |\gamma_k|^2$$

FOR PHYSICISTS...

OBSERVABLE := Hermitian operator. $\hat{O}$

(By SPECTRAL THEOREM) O (1) - Has eigenbasis $\{|\psi_i\rangle\}_i$

(2) - eigenvectors λ_i $\left(O|\psi_i\rangle = \lambda_i |\psi_i\rangle\right)$

(1)+(2) => defines a basis $\{|\psi_i\rangle\}$, indexed by $\underline{\underline{\lambda_i}}$

=> "outcomes"

- DEGENERACY
- EXPECTED VALUE (EXPECTATION)

} TUTORIALS / TAKE HOME ASSIGNMENT

- MEASUREMENT OF SUBSYSTEM & POST-MEASUREMENT STATE => TUTORIALS

POSTULATE 4 COMPOSITE SYSTEM

Let S_1 & S_2 BE TWO Q.M. SYSTEMS

WITH $\mathbb{C}^N$, $\mathbb{C}^M$ AS STATE SPACES

THEN THE COMPOSITE SYSTEM S_{12} HAS STATE SPACE

$$\mathbb{C}^N \otimes \mathbb{C}^M \simeq \mathbb{C}^{N \times M}$$

↑

TENSOR PRODUCT...

THE MATH

PRODUCT... IMPORTANT

$$\otimes : \mathbb{C}^N \times \mathbb{C}^M \mapsto \mathbb{C}^{N \times M} \quad , \quad \text{bilinear}$$

$$|\psi\rangle \otimes |\varphi\rangle$$

ON MATRICES (NUMERICAL VECTORS): KRONECKER PRODUCT

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \otimes \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_M \end{bmatrix} = \begin{bmatrix} \alpha_1 \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_M \end{pmatrix} \\ \vdots \\ \alpha_N \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_M \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \alpha_1 \beta_1 \\ \alpha_1 \beta_2 \\ \vdots \\ \alpha_1 \beta_M \\ \alpha_2 \beta_1 \\ \vdots \\ \alpha_N \beta_M \end{bmatrix} ;$$

$$\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1N} \\ \vdots & & \\ \alpha_{N1} & \cdots & \alpha_{NN} \end{bmatrix} \otimes B = \begin{bmatrix} \alpha_{11} B & \alpha_{12} B & \cdots & \alpha_{1N} B \\ \vdots & & & \\ \alpha_{N1} B & & \cdots & \alpha_{NN} B \end{bmatrix}$$

$$A \otimes (B + C) = A \otimes B + A \otimes C$$
$$(B + C) \otimes A = B \otimes A + C \otimes A$$
$$(k A) \otimes B = k (A \otimes B)$$

ASSOCIATIVE

$$A \otimes 0 = 0 \otimes A = 0$$

USEFUL $U_1 \otimes U_2$ is UNITARY...

SPECTRAL THEO ON $M_1 \otimes M_2$?

QUANTUM COMPUTING VULGARIS...

QUBIT = S WITH $\mathbb{C}^2$ = "TWO (BASIS) STATES" $|0\rangle, |1\rangle$

<u>REGISTER</u> : $\underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n} = (\mathbb{C}^2)^{\otimes n}$

$$\begin{array}{lll} & (b_n \ldots b_0)_2 & (int) \\ |0\rangle \otimes |0\rangle \cdots |0\rangle \otimes |0\rangle & \equiv |0 \ldots 0\rangle & = |0\rangle \\ |0\rangle \otimes |0\rangle \cdots |0\rangle \otimes |1\rangle & \equiv |0 \ldots 1\rangle & |1\rangle \\ \vdots & & \vdots \\ |1\rangle \otimes |1\rangle \cdots |1\rangle \otimes |1\rangle & \equiv |111 \ldots 1\rangle & = |2^n - 1\rangle \end{array}$$

<u>STATE</u> : $|\psi^{(n)}\rangle = \sum_{b_n \ldots b_1 = 0,0,\ldots 0}^{1,1,\ldots 1} \underbrace{\alpha_{b_n b_{n-1} \ldots b_1}} |b_n b_{n-1} \ldots b_1\rangle$; $\sum_{\vec{b}} |\alpha_{\vec{b}}|^2 = 1$,

$\alpha_{\vec{b}} \in \mathbb{C}$

exponentially many complex numbers

$$\begin{bmatrix} \alpha_{0 \ldots 0} \\ \alpha_{0 \ldots 01} \\ \alpha_{0 \ldots 10} \\ \vdots \\ \vdots \\ \alpha_{1111} \end{bmatrix} \begin{array}{l} \leftarrow 0^{th} \\ \leftarrow 1^{st} \\ \leftarrow 2^{nd} \\ \\ \\ \leftarrow (2^{n}-1)^{st} \end{array}$$

Product state : $|\psi_{AB}\rangle = |\psi_A\rangle |\psi_B\rangle$; No? ENTANGLED.

ENTANGLING GATE = NOT EXPRESSIBLE AS PRODUCT $U_1 \otimes U_2$ (CNOT)

$\Rightarrow$ INTRODUCES CORRELATIONS

QUANTUM COMPUTING VULGARIS...

- EVOLUTION OF REGISTER : ANY <u>UNITARY MAP</u> U

- READOUT : COMPUTATIONAL BASIS $\{|0..0\rangle, |00..1\rangle, |0..10\rangle .., |1..1\rangle\}$
  MEASUREMENT
  (RECALL : THIS + <u>BASIS CHANGE</u>

$\rightarrow$ PROBABILITY OF $(b_n b_{n-1} .. b_1)$,

BORN RULE : $P(b_n .. b_1) = |\underbrace{\langle b_n .. b_1|}_{\text{Possible output.}} \underbrace{U}_{\text{"computation"}} \underbrace{|00..00\rangle}_{\text{initial}}|^2$

REGISTER ✓

READOUT ✓

EVOLUTION V.S. GATE - SET ?

- QUANTUM GATE SET & QUANTUM CIRCUITS

$$\text{GATE SET} = \{ U_j \mid j \in \mathbb{I},\ U_j \text{ is unitary} \}$$

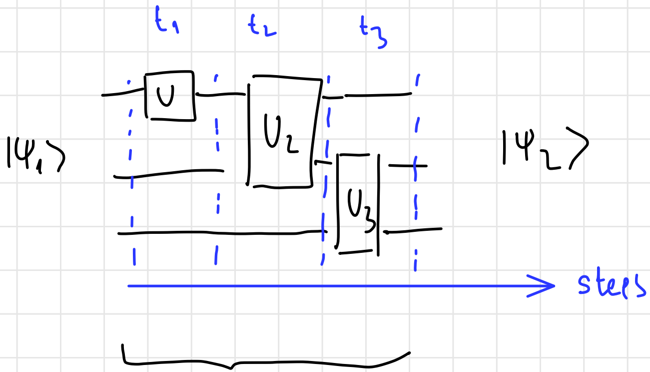
- <u>LOCAL UNITARY</u>

Eg. $U \in (\mathbb{C}^{2\times 2}) \Rightarrow U = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$; $\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$|\psi\rangle = \overbrace{|0\rangle|0\rangle \cdots |0\rangle}^{n} \quad \left(= |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle = |000\cdots 0\rangle \right)$$

$$U^{(k)}|\psi\rangle = \mathbb{1}_2^{\otimes(k-1)} \otimes U \otimes \mathbb{1}_2^{\otimes(n-k-1)} |\psi\rangle$$

QUANTUM CIRCUIT



TENSOR PRODUCT (DON'T FORGET $\mathbb{1}$)

OPERATOR COMPOSITION (MATRIX PRODUCT)

$\mathcal{C}$: pictorial representation of some Unitary

$$|\psi_2\rangle = (\mathbb{1}_2 \otimes U)(U_2 \otimes \mathbb{1}_2)(U \otimes \mathbb{1}_2^{\otimes 2})|\psi_1\rangle$$

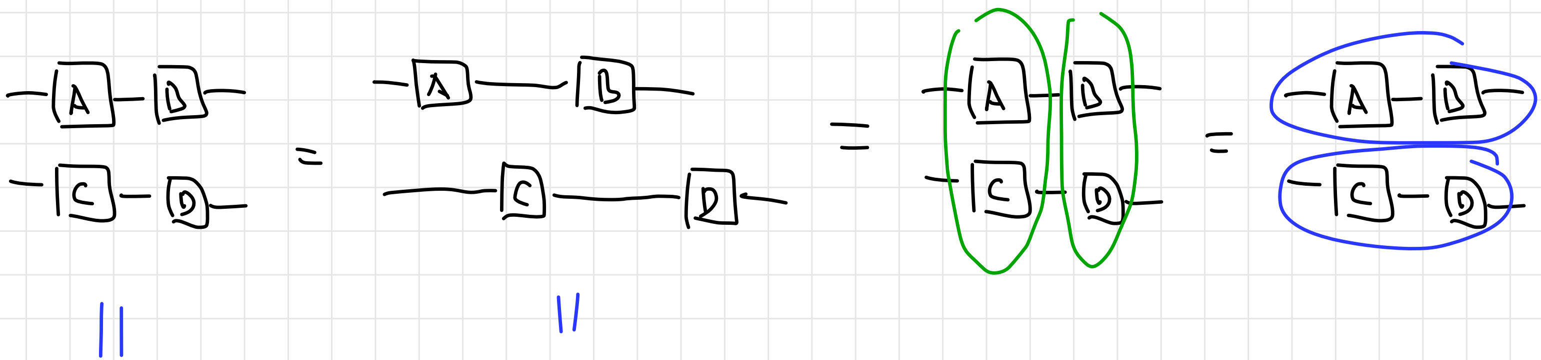
steps

t_3 t_2 t_1

KNOW HOW TO COMPUTE THE MATRIX (ENTIRE!) GIVEN A CIRCUIT.

QUANTUM CIRCUIT (1)

Implicit algebra



$$(B\otimes D)(A\otimes C) = (\mathbb{1}\otimes D)(B\otimes \mathbb{1})(\mathbb{1}\otimes C)(A\otimes \mathbb{1}) = (B\otimes D)(A\otimes C) = (B\cdot A)\otimes(D\cdot C)$$

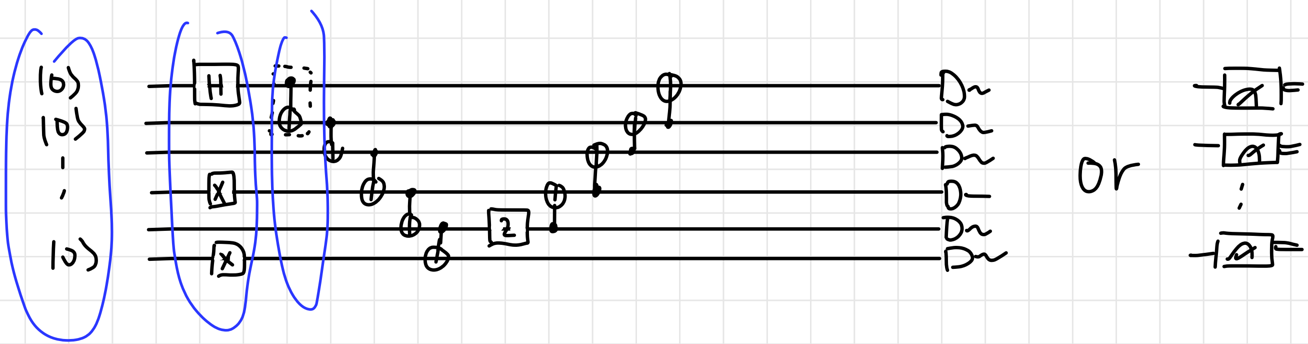
ALL THEOREMS ... IMPLICIT IN

CIRCUIT FORMALISM

1.) initial state

2.) sequence of gates from Gate set

3.) measurement



2^n-vector

$2^n \times 2^n$ matrices

SOME SPECIAL GATES: SINGLE QUBIT GATES

RECALL: $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

FOR PHYSICISTS

PAULI OPERATORS (GATES)

$$— = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma^x \quad -\boxed{X}- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \iff X|b\rangle = |1 \oplus b\rangle \implies \text{"NOT"}$$

$$\sigma^z \quad -\boxed{Z}- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \iff Z|b\rangle = (-1)^b |b\rangle$$

$$\sigma^y \quad -\boxed{Y}- = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y|b\rangle = i(-1)^b |b \oplus 1\rangle$$

XOR or $+_{\text{mod } 2}$

HADAMARD

$$-\boxed{H}- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H|b\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + (-1)^b |1\rangle \right)$$

"$\pi/8$"

$$-\boxed{\pi/8}- = -\boxed{T}- = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

"Z-rotation"

$$-\boxed{Z_\phi}- = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

CHECK:

$$HZH = ?$$

READ: PAULI COMMUTATION RELATIONS

$$[A,B] = AB - BA \qquad (A, B - \text{Pauli})$$

MOST GENERAL:

$$SU(2) \ni U = e^{i\alpha} \begin{bmatrix} \cos\theta & -e^{i\lambda} \sin\theta \\ e^{i\phi} \sin\theta & e^{i(\phi+\lambda)} \cos\theta \end{bmatrix}$$

$$= e^{i\alpha} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{bmatrix}$$

$$= e^{i\alpha} Z_{\phi'} \underbrace{H Z_{\theta'} H}_{X_{\theta'} \text{ (UP TO PHASES)}} Z_{\lambda'}$$

GLOBAL PHASE

"GLOBAL PHASE IS IRRELEVANT"

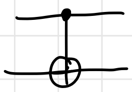
$$|\langle \psi_{out}| U_2 U_1 U_0 |\psi_{in}\rangle|^2 = |\langle \psi_{out}| U_2 (\exp(i\theta) U_1) U_0 |\psi_{in}\rangle|^2$$

$\theta \in \mathbb{R}$

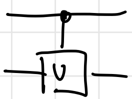
For all $U_0, U_1, U_2, \theta \ldots$

DEEP WATER...

Some special gates: Two qubit gates (Entangling gates...)

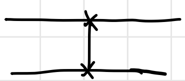


$$= \begin{bmatrix} \mathbb{1}_2 & \bar{0} \\ \bar{0} & X \end{bmatrix} = \text{CNOT}\,|b_1 b_2\rangle = |b_1\ b_1 \oplus b_2\rangle$$



$$= \begin{bmatrix} \mathbb{1} & \bar{0} \\ \bar{0} & U \end{bmatrix} = C\text{-}U\,|b_1 b_2\rangle = |b_1\rangle\, U^{b_1} |b_2\rangle$$

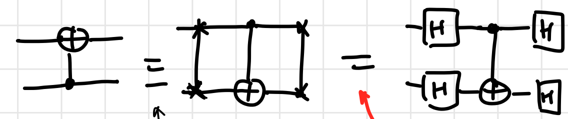
$$(U^0 = \mathbb{1})$$



$$= \text{SWAP}\,|b_1 b_2\rangle = |b_2\ b_1\rangle = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{X}} & 0 \\ 0 & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

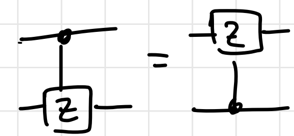
↳ wierd one...

Some properties



can take as definition

Prove



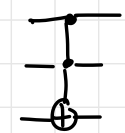
Physically motivated

- Mølmer - Sørensen : $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}$
- $i\text{SWAP}(t) = \exp(-iHt)$; $H = -X \otimes X - Y \otimes Y$
- $C\text{-}Z_\phi$

Discussed in a few lectures

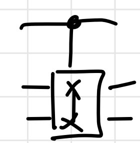
Three qubit:

Toffoli :



ctrl-ctrl-X

Fredkin :



ctrl-SWAP

Two universality results

$$G_1 = \{CNOT\} \cup SU(2)$$

all single qubit gates

Th 1. G_1 is exactly universal i.e.

Every n-qubit unitary can exactly be decomposed into $2^{O(n)}$ CNOT & single-qubit gates...

COMPARE TO CLASSICAL UNIVERSALITY

Solovay-Kitaev: $G_2 = \{H, \pi/8\}$; any single qubit unitary can be exponentially efficiently approximated using just G_2

Cor. $G_3 = \{CNOT, H, \pi/8\}$ is approximately universal

FUNKY ADVANCED RESULT...

TOFFOLI IS UNIVERSAL FOR CLASSICAL COMPUTING

$\underbrace{\text{TOFFOLI} + H}$ IS APPROXIMATELY COMPUTATIONALLY UNIVERSAL FOR QC.

ONLY REAL MATRICES...

CANNOT BE DONE WITH 2-qubit REAL GATES...

"COMPUTATIONALLY UNIVERSAL" = IF "STANDARD" QC CAN COMPUTE SOME FUNCTION EFFICIENTLY, SO CAN TOFFOLI + H Q.C.

- CLEARLY CANNOT PREPARE STATES WITH COMPLEX AMPLITUDES...

JJ

QUANTUM ALGORITHM

FOR PROBLEM/TASK T IS A PERSCRIPTION / TURING MACHINE THAT GIVEN INPUT SIZE n OUTPUTS QUANTUM CIRCUIT $\mathcal{C}_n$ ON N qubits...

TYPICALLY A Q ALGO INVOLVES MEASUREMENT IN CLASSICAL (COMPUTATIONAL) BASIS + REPETITION & POST-PROCESSING

MUST KNOW: GIVEN A CIRCUIT & INITIAL STATE, COMPUTE OUTPUT STATE, AND MEASUREMENT PROBABILITIES