Outline

This assignment consists of 12 questions, each carrying 1 or 2 points (16 in total). It constitutes 50% of your total take-home assignment mark (25% of the overall mark for this course).

Deadline: the beginning of the lecture 28th of February 2020.

Math background

1. (1 point) Consider the following two vectors:

$$|\psi\rangle = \begin{bmatrix} 1 \\ -1 \\ i \\ \exp(i3\pi/2) \end{bmatrix}, |\phi\rangle = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ \exp(i\pi/2) \\ i \end{bmatrix} \tag{1}$$

   where $i = \sqrt{-1}$.

   Compute their (a) sum, (b) inner product $\langle\psi|\phi\rangle$, and (c) the Euclidean norm of each; (d) are they orthogonal, orthonormal or neither? Be careful, the inner product over complex spaces is sesquilinear.

2. (1 point) Prove that matrix-vector multiplication is a linear operation in two dimensions, i.e. prove the following holds by explicitly computing the products:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) + \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right) \tag{2}$$

   Naturally, the same property holds for all dimensions.

3. (2 points) Consider the following two matrices:

$$V = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; W = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3}$$

   Answer the following: (a) do V and W commute; (b) are they unitary, (c) are they hermitian; (d) what is their trace; (e) what is their rank; (f) provide the spectral decomposition of V and W; (g) what are their eigenvectors and eigenvalues?

4. (1 point) Provide the Taylor series expansion for $exp(x)$, $sin(x)$ and $cos(x)$. Using these (and absolute convergence properties) show that $exp(ix) = cos(x) + i\ sin(x)$.

Postulates of quantum mechanics

5. (1 point) (a) Specify the state space (Hilbert space) of a system of $n = 3$ qubits; (b) provide the general state vector of a (pure) quantum state in that space, and (c) explicitly specify the normalization constraint.

6. (2 points) Let $H = V + \mathbf{1}$, where $\mathbf{1}$ is the identity matrix, and V the 2×2 matrix from Question 3. (a) Specify the evolution of the system under H for time t. (b) Check that the time-integrated evolution $U(t)$ is unitary for all t. You can use the spectral decomposition of H to solve this. N.B.: in the following questions you can work in "natural units", i.e., $\hbar = 1$.

7. (2 points) Let H as given in the above question be an observable.

   (a) Compute the spectral decomposition of H, and identify the eigenvectors $|\lambda_1\rangle$ and $|\lambda_2\rangle$, and the corresponding eigenvalues λ_1, λ_2.

   Consider the state $|\psi\rangle = U(\pi)|0\rangle$ for U as defined in Question 6. Note, now the state $|\psi\rangle$ can be explicitly computed, as all matrices are known.

   (b) Suppose we measure the observable H on the quantum state $|\psi\rangle$. Compute the outcome probabilities p_{λ_1} and p_{λ_2}, associated to the eigenvalues of H explicitly.

   (c) Compute the expectation value $\langle\psi|H|\psi\rangle$ explicitly. Express the expectation value in terms of the probabilities $p_{\lambda_1}, p_{\lambda_2}$ and the eigenvalues.

8. (1 point) Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, |\psi\rangle = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, |\phi\rangle = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

   Prove that $(A \otimes B)(|\psi\rangle \otimes |\phi\rangle) = (A\,|\psi\rangle) \otimes (B\,|\phi\rangle)$, by explicitly computing the tensor and matrix products.

9. (1 point) Is the tensor product commutative? Prove or give a counterexample.

10. (1 point) If A has eigenvalues λ_j and corresponding eigenvectors $|\lambda_j\rangle$, and B has eigenvalues ω_j and corresponding eigenvectors $|\omega_j\rangle$, what are the eigenvalues and eigenvectors of $A \otimes B$? Use the results of the previous question.

11. (1 point) Write down the commutation relations $[P_1, P_2]$ for the three Pauli matrices $P_i \in \{X, Y, Z\}$. (No need to compute the symmetric cases e.g. if you compute $[X, Y]$ no need to provide $[Y, X]$)

12. (2 points) Prove that $e^{i\theta P} = \cos(\theta)I + i\sin(\theta)P$ for a Pauli operator P. Use the results of Question 4.