Quantum Algorithms tutorial

Hamiltonian simulation and the HHL algorithm

Casper Gyurik

Leiden Institute of Advanced Computer Science

November 7th, 2019

Universiteit Leiden
The Netherlands

Hamiltonian simulation

HHL

Hamiltonian simulation

What does it do?

Hamiltonian simulation (HS)

Given *access* to a Hermitian matrix $H : (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$ and some $t > 0$.
Construct a quantum circuit that implements the unitary e^{iHt}.

Hamiltonian simulation

What does it do?

Hamiltonian simulation (HS)

Given *access* to a Hermitian matrix $H : \left(\mathbb{C}^2\right)^{\otimes n} \to \left(\mathbb{C}^2\right)^{\otimes n}$ and some $t > 0$.
Construct a quantum circuit that implements the unitary e^{iHt}.

- Have to consider special access, since H is an exponentially large matrix.
  - k-local $H = \sum_j H_j$, where each H_j acts nontrivially on at most k qubits.
  - Sparse oracle access, i.e., can query nonzero entries of each collumn.

Hamiltonian simulation

What does it do?

Hamiltonian simulation (HS)

Given *access* to a Hermitian matrix $H : (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$ and some $t > 0$. Construct a quantum circuit that implements the unitary e^{iHt}.

- Have to consider special access, since H is an exponentially large matrix.
  - k-local $H = \sum_j H_j$, where each H_j acts nontrivially on at most k qubits.
  - Sparse oracle access, i.e., can query nonzero entries of each collumn.
- For any Hermitian H and $t > 0$, the operator

$$e^{iHt} = \sum_{j=0}^{\infty} \frac{(iHt)^j}{j!}$$

is unitary $\implies$ it can be implemented as a quantum circuit.

Hamiltonian simulation

What does it do?

Hamiltonian simulation (HS)

Given *access* to a Hermitian matrix $H : \left(\mathbb{C}^2\right)^{\otimes n} \to \left(\mathbb{C}^2\right)^{\otimes n}$ and some $t > 0$. Construct a quantum circuit that implements the unitary e^{iHt}.

- Have to consider special access, since H is an exponentially large matrix.
  - k-local $H = \sum_j H_j$, where each H_j acts nontrivially on at most k qubits.
  - Sparse oracle access, i.e., can query nonzero entries of each collumn.
- For any Hermitian H and $t > 0$, the operator

$$e^{iHt} = \sum_{j=0}^{\infty} \frac{(iHt)^j}{j!}$$

is unitary $\implies$ it can be implemented as a quantum circuit.

- In physics Hermitian matrices represent Hamiltonians which, via the Schrödinger equation, tell you how a quantum system evolves over time.

Hamiltonian simulation

What does it do?

Hamiltonian simulation (HS)

Given *access* to a Hermitian matrix $H : \left(\mathbb{C}^2\right)^{\otimes n} \to \left(\mathbb{C}^2\right)^{\otimes n}$ and some $t > 0$. Construct a quantum circuit that implements the unitary e^{iHt}.

- Have to consider special access, since H is an exponentially large matrix.
  - k-local $H = \sum_j H_j$, where each H_j acts nontrivially on at most k qubits.
  - Sparse oracle access, i.e., can query nonzero entries of each collumn.
- For any Hermitian H and $t > 0$, the operator

$$e^{iHt} = \sum_{j=0}^{\infty} \frac{(iHt)^j}{j!}$$

  is unitary $\implies$ it can be implemented as a quantum circuit.
- In physics Hermitian matrices represent Hamiltonians which, via the Schrödinger equation, tell you how a quantum system evolves over time.

Exercise: how to use HS to sample from eigenvalues of a *Hermitian* matrix.

The HHL algorithm

What does it do?

The HHL algorithm solves the so-called "quantum linear system" problem.

The quantum linear system (QLS) problem

Given some quantum state $|b\rangle \propto \sum_{i=1}^{N} b_i |i\rangle$ and sparse access to $A \in \mathbb{C}^{N \times N}$.
Prepare the quantum state $|x\rangle \propto A^{-1} \sum_{i=1}^{N} b_i |i\rangle$.

The HHL algorithm

What does it do?

The HHL algorithm solves the so-called "quantum linear system" problem.

The quantum linear system (QLS) problem

Given some quantum state $|b\rangle \propto \sum_{i=1}^{N} b_i |i\rangle$ and sparse access to $A \in \mathbb{C}^{N \times N}$.
Prepare the quantum state $|x\rangle \propto A^{-1} \sum_{i=1}^{N} b_i |i\rangle$.

- Here the symbol $\propto$ stands for "proportional to"
  - **Note:** $\sum_{i=1}^{N} b_i |i\rangle$ is not necessarily a unit vector.

The HHL algorithm

What does it do?

The HHL algorithm solves the so-called "quantum linear system" problem.

The quantum linear system (QLS) problem

Given some quantum state $|b\rangle \propto \sum_{i=1}^{N} b_i |i\rangle$ and sparse access to $A \in \mathbb{C}^{N \times N}$.
Prepare the quantum state $|x\rangle \propto A^{-1} \sum_{i=1}^{N} b_i |i\rangle$.

- Here the symbol $\propto$ stands for "proportional to"
  - **Note:** $\sum_{i=1}^{N} b_i |i\rangle$ is not necessarily a unit vector.
- We say $|b\rangle$ is an "amplitude encoding" of $\vec{b} = (b_i)_{i=1}^{N} \in \mathbb{C}^N$.
- Likewise, we prepare $|x\rangle$ which is an "amplitude encoding" of $\vec{x} = A^{-1}\vec{b}$.

The HHL algorithm

What does it do?

The HHL algorithm solves the so-called "quantum linear system" problem.

The quantum linear system (QLS) problem

Given some quantum state $|b\rangle \propto \sum_{i=1}^{N} b_i |i\rangle$ and sparse access to $A \in \mathbb{C}^{N \times N}$.
Prepare the quantum state $|x\rangle \propto A^{-1} \sum_{i=1}^{N} b_i |i\rangle$.

- Here the symbol $\propto$ stands for "proportional to"
  - **Note:** $\sum_{i=1}^{N} b_i |i\rangle$ is not necessarily a unit vector.
- We say $|b\rangle$ is an "amplitude encoding" of $\vec{b} = (b_i)_{i=1}^{N} \in \mathbb{C}^N$.
- Likewise, we prepare $|x\rangle$ which is an "amplitude encoding" of $\vec{x} = A^{-1}\vec{b}$.
- Note that this is different from the "classical linear system" problem.
  - Have to prepare the input $|b\rangle$, how to do it from classical description of $\vec{b}$?
  - Don't get a classical description of $\vec{x}$, how to get it from $|x\rangle$?

The HHL algorithm

How does it work? A brief refresher on "spectral decompositions".

To understand how HHL works, one should consider "spectral decompositions".

The HHL algorithm

How does it work? A brief refresher on "spectral decompositions".

To understand how HHL works, one should consider "spectral decompositions".

Theorem (Spectral theorem for Hermitian matrices)

Any Hermitian matrix $A \in \mathbb{C}^{N \times N}$ *has* N *orthogonal eigenvectors* $\{|\psi_j\rangle\}_{j=1}^{N}$ *that form a basis of* $\mathbb{C}^N$*.*

The HHL algorithm

How does it work? A brief refresher on "spectral decompositions".

To understand how HHL works, one should consider "spectral decompositions".

Theorem (Spectral theorem for Hermitian matrices)

Any Hermitian matrix $A \in \mathbb{C}^{N \times N}$ *has* N *orthogonal eigenvectors* $\{|\psi_j\rangle\}_{j=1}^{N}$ *that form a basis of* $\mathbb{C}^N$*.*

- Moreover, one can write $H = \sum_{j=1}^{N} \lambda_j |\psi_j\rangle \langle\psi_j|$, where λ_j denotes the eigenvalue of $|\psi_j\rangle$.
  - This is often called the "spectral decomposition" of H.

The HHL algorithm

How does it work? A brief refresher on “spectral decompositions”.

To understand how HHL works, one should consider “spectral decompositions”.

Theorem (Spectral theorem for Hermitian matrices)

Any Hermitian matrix $A \in \mathbb{C}^{N \times N}$ has N orthogonal eigenvectors $\{|\psi_j\rangle\}_{j=1}^N$ that form a basis of $\mathbb{C}^N$.

- Moreover, one can write $H = \sum_{j=1}^N \lambda_j |\psi_j\rangle \langle\psi_j|$, where λ_j denotes the eigenvalue of $|\psi_j\rangle$.
  - This is often called the “spectral decomposition” of H.
- Since $\{|\psi_j\rangle\}_{j=1}^N$ is a basis of $\mathbb{C}^N$, any vector $|v\rangle \in \mathbb{C}^N$ can be written as

$$|v\rangle = \sum_{j=1}^N \alpha_j |\psi_j\rangle .$$

  - This is often called the “spectral decomposition” of $|v\rangle$ w.r.t. H.

The HHL algorithm

How does it work? A brief refresher on "spectral decompositions".

To understand how HHL works, one should consider "spectral decompositions".

Theorem (Spectral theorem for Hermitian matrices)

Any Hermitian matrix $A \in \mathbb{C}^{N\times N}$ has N orthogonal eigenvectors $\{|\psi_j\rangle\}_{j=1}^N$ that form a basis of $\mathbb{C}^N$.

- Moreover, one can write $H = \sum_{j=1}^N \lambda_j |\psi_j\rangle \langle\psi_j|$, where λ_j denotes the eigenvalue of $|\psi_j\rangle$.
  - This is often called the "spectral decomposition" of H.
- Since $\{|\psi_j\rangle\}_{j=1}^N$ is a basis of $\mathbb{C}^N$, any vector $|v\rangle \in \mathbb{C}^N$ can be written as

$$|v\rangle = \sum_{j=1}^{N} \alpha_j |\psi_j\rangle .$$

  - This is often called the "spectral decomposition" of $|v\rangle$ w.r.t. H.

Exercise: Show that $A^{-1}|v\rangle = \sum_{j=1}^N \lambda_j^{-1}\alpha_j |\psi_j\rangle$.
Remark: HHL solves QLS problem using the above decomposition, HS & QPE.