Quantum Algorithms tutorial 3

The Deutsch-Jozsa algorithm

Casper Gyurik

Leiden Institute of Advanced Computer Science

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Universiteit Leiden
The Netherlands

The Deutsch-Jozsa problem

An example of an oracle problem

The Deutsch-Jozsa problem

Input: Oracle access to a function $f : \{0,1\}^n \to \{0,1\}$ which you are *promised* either satisfies

1. f is constant, i.e., $f(j) = 1$ for all j or $f(j) = 0$ for all j.
2. f is balanced, i.e., $|f^{-1}(1)| = \frac{2^n}{2}$.

Problem: Decide whether case 1. or case 2. holds.

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Definition (Oracle acces to a function f)

- Classically, oracle access to a function f allows you to compute the map

$$\text{Query}_f : j \mapsto f(j), \text{ for } j \in \{0,1\}^n.$$

- In the quantum setting, oracle access is given by the unitary

$$O_f : |j\rangle |b\rangle \mapsto |j\rangle |b \oplus f(j)\rangle, \text{ for } j \in \{0,1\}^n \text{ and } b \in \{0,1\}.$$

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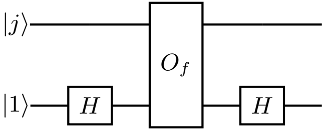
The phase oracle

Using the oracle O_f we can implement the phase oracle $O_{f,\pm}$ that maps

$$O_{f,\pm} : |j\rangle \mapsto (-1)^{f(j)} |j\rangle ,$$



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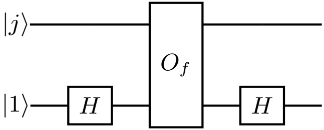

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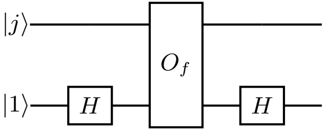

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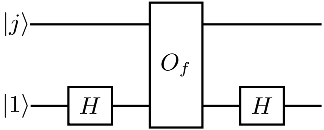

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Problem: Decide whether case 1. or case 2. holds.

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5. Measure the system in the computational basis.

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When measuring the output of the Deutsch-Jozsa algorithm we have

$$\Pr(0^n) = \begin{cases} 1 \text{ if } f \text{ is constant} \\ 0 \text{ if } f \text{ is balanced} \end{cases} .$$

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Evaluating the circuit

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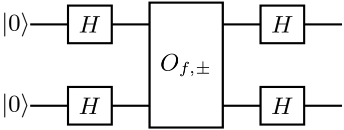
The Deutsch-Jozsa algorithm for $n = 2$ is given by the following circuit



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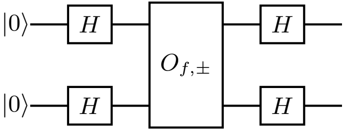

Exercise: 1. Compute the output of the above circuit for arbitrary f.
2. What is the output if f satisfies the Deutsch-Jozsa promise?