Quantum Algorithms tutorial 2

Postulates of quantum mechanics and circuit evaluation

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Postulates of quantum mechanics

Refresher: how to work with different bases

Evaluating a quantum circuit

Solovay Kitaev theorem

The postulates of quantum mechanics

What are the rules of quantum computing?

Postulate 1: State space

Associated to any physical system is a complex vector space $\mathbb{C}^N$ called the *state space*. The system is completely described by the *state vector*, which is a unit vector in the state space $|\psi\rangle \in \mathbb{C}^N$.

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- A qubit system, who is completely described by the state vector

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- An n-qubit system, which is completely described by the state vector

$$|\psi\rangle = \sum_{j\in\{0,1\}^n} \alpha_j\,|j\rangle \in \left(\mathbb{C}^2\right)^{\otimes n} \cong \mathbb{C}^{2^n}.$$

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Characterizations of unitary matrices

- U is unitary $\iff$ U is norm-preserving.
- U is unitary $\iff$ the columns of U form orthonormal basis for $\mathbb{C}^n$.
- U is unitary $\iff$ $U^\dagger U = UU^\dagger = I$, where I is the identity and $U^\dagger = \overline{U}^T$.

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Exercise: U norm-preserving $\implies$ U is invertible.

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More on the postulates in N&C chapters 2.2.1-2.2.3

Working with different bases

Expressing vectors in different bases

For single qubit states, we know the following two bases:

Computational basis:

$$\{|0\rangle, |1\rangle\}.$$

Hadamard basis:

$$\{|+\rangle, |-\rangle\}.$$

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- We can rewrite this vector *in the Hadamard basis* as

$$|\psi\rangle = \alpha_+ |+\rangle + \alpha_- |-\rangle .$$

where we can compute the values α_+ and α_- using

$$\alpha_+ = \langle + \mid \psi\rangle \text{ and } \alpha_- = \langle - \mid \psi\rangle .$$

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Example

Consider $|\psi\rangle = |1\rangle$, then $\langle + \mid \psi\rangle = \frac{1}{\sqrt{2}}$ and $\langle - \mid \psi\rangle = -\frac{1}{\sqrt{2}}$ and thus
$$|1\rangle = \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle .$$

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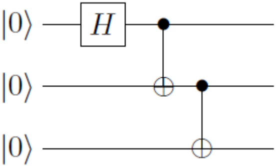
Exercise: rewrite $|\psi\rangle = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$ in the Hadamard basis.

Evaluating a quantum circuit

How do we compute its output?



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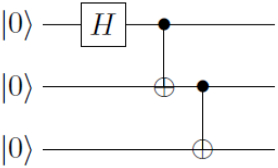

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After first layer of gates:

$$\begin{aligned} |\psi\rangle &= (H \otimes I \otimes I)(|0\rangle \otimes |0\rangle \otimes |0\rangle) = H\,|0\rangle \otimes |0\rangle \otimes |0\rangle \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |100\rangle) \end{aligned}$$

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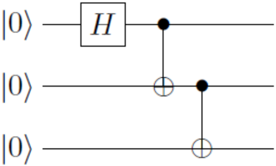

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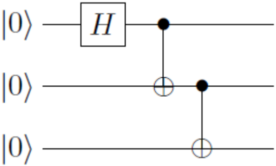

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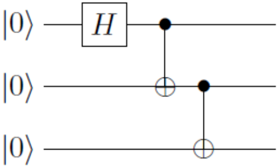

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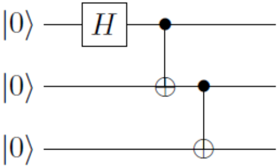

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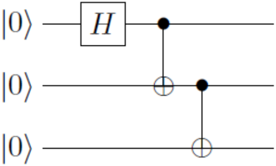

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More on quantum circuits in N&C chapters 1.2-1.3, 4.1-4.4 and 4.6

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Moreover, it turns out that the set $\{H, R_{\pi/8}\}$ is universal for single qubit gates.