→ STATE OF n- QUBIT REGISTER DESCRIBED BY 2^n complex numbers

ACCESSING INFORMATION?

Q: HOW DO WE ACCESS THE INFO OF n- COIN STOCHASTIC SYSTEM?

→ description ALSO exponential... 2^n probabilities?

description ... vs .. state

QUANTUM MEASUREMENT

1.) "computational basis measurement"

$$|\psi\rangle = \sum_{b_1 \ldots b_n} \alpha_{b_1 \ldots b_n} |b_1 \ldots b_n\rangle ;$$

$$|\psi\rangle \longrightarrow \boxed{M} \longrightarrow \text{outcome } b_1 \ldots b_n$$

$$\boxed{P_M\left(b_1 \ldots b_n \,\middle|\, |\psi\rangle\right) = |\alpha_{b_1 \ldots b_n}|^2}$$

"observation of Q. system"

... "observation of hidden coins"

QUANTUM MEASUREMENT

1.) "computational basis measurement"

$$|\psi\rangle = \sum_{b_1 \dots b_n} \alpha_{b_1 \dots b_n} |b_1 \dots b_n\rangle ;$$

$$|\psi\rangle = \begin{array}{c l} \begin{bmatrix} \alpha_{0 \dots 0} \\ \vdots \\ \alpha_{1 \dots 1} \end{bmatrix} & \begin{array}{l} \leftarrow 000 \dots 0 \\ \leftarrow 00 \dots 1 \\ \vdots \\ \leftarrow 11111 \end{array} \end{array}$$

$$|\psi\rangle \text{ --- } \boxed{M} = \text{ outcome } b_1 \dots b_n$$

NORM-SQUARED ...

"observation of Q. system"

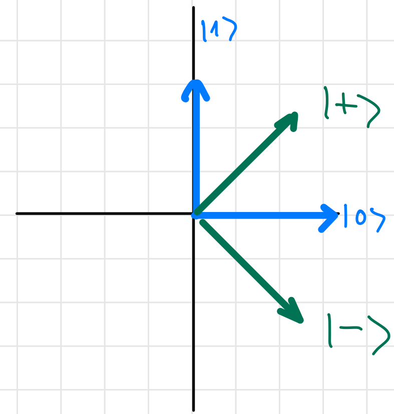
... "observation of hidden coins"

Recall normalization

MANY TYPES OF MEASUREMENTS..

→ COMPLETE PROJECTIVE MEASUREMENTS

→ SPECIFIED BY A BASIS OF ~~HILBERT~~ VECTOR SPACE



$\{|0\rangle, |1\rangle\}$;

$$|+\rangle := \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|-\rangle := \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$\mathrm{span}_{\mathbb{C}}\{|0\rangle, |1\rangle\} = \mathrm{span}_{\mathbb{C}}\{|+\rangle, |-\rangle\}$$

"C"-basis   "H"-basis

Z-basis   X-basis

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$\tilde{b} \in \{0,1\}$

$0 \to$ "$|+\rangle$"

$1 \to$ "$|-\rangle$"

What is

$$P(\tilde{b} \,|\, |\psi\rangle, \text{"}H\text{"}) = ?$$

Method a) express $|\psi\rangle$ as linear combination of $|+\rangle$, $|-\rangle$, and as usual

EQUIVALENTLY

b) inner products

$$(|\psi\rangle, |\varphi\rangle) = \sum_i \bar{\alpha}_i \beta_i$$

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \cdot \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} \bar{\alpha}_1 & \bar{\alpha}_2 & \dots & \bar{\alpha}_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

What is

$$P(\tilde{b} \,|\, |\psi\rangle, \text{"H"}) = ?$$

$$|\psi\rangle - \boxed{M_H} \overset{?}{=} \tilde{b}$$

Method a) express $|\psi\rangle$ as linear combination of $|+\rangle$, $|-\rangle$, and as usual

$\tilde{b} \in \{0,1\}$

$0 \to$ "$|+\rangle$"

$1 \to$ "$|-\rangle$"

EQUIVALENTLY

b) inner products

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Q.M. notation for inner product

$$(|\psi\rangle, |\varphi\rangle) =: \boxed{\langle \psi | \varphi \rangle \in \mathbb{C}}$$

$$P(\tilde{b}=0 \mid |\psi\rangle, H) = |\langle + | \psi \rangle|^2$$

$$P(\tilde{b}=1 \mid |\psi\rangle, H) = |\langle - | \psi \rangle|^2$$

ordering matters.

choose $0 \Leftrightarrow |+\rangle$

$1 \Leftrightarrow |-\rangle$

because Hadamard

EXAMPLE INNER PRODUCT

bilinearity ..

(actually sesquilinearity

as $(\alpha|\psi\rangle, |\varphi\rangle) \to \bar{\alpha}(|\psi\rangle, |\varphi\rangle)$)

$$|\psi_1\rangle = \alpha|+\rangle + \beta|-\rangle$$

$$|\psi_2\rangle = \gamma|0\rangle + \delta|1\rangle$$

$$\langle \psi_1 | \psi_2 \rangle = (\bar{\alpha}\langle +| + \bar{\beta}\langle -|)(\gamma|0\rangle + \delta|1\rangle)$$

$$= \bar{\alpha}\gamma\underbrace{\langle + | 0 \rangle}_{1/\sqrt{2}} + \cdots \quad \bar{\beta}\delta \underbrace{\langle - | 1 \rangle}_{-\frac{1}{\sqrt{2}}}$$

Born rule of QUANTUM MECHANICS

$M \Leftarrow \{|\varphi_1\rangle \dots |\varphi_{2^n}\rangle\}$, $O = \{1, \dots, 2^n\}$

in : $|\psi\rangle$

$$P(k \mid |\psi\rangle, M) = |\langle \varphi_k | \psi \rangle|^2$$

Q: NORMALIZATION ?

Q: MEASURING <u>ONLY</u> <u>SOME</u> QUBITS?

MANIPULATING QUANTUM REGISTERS

IN ANALOGY TO CLASSICAL CIRCUITS

"LEGAL" OPERATIONS

CLASSICAL : ANY $f : \{0,1\}^n \rightarrow \{0,1\}^m$

PROBABILISTIC CLASSICAL: ANY LINEAR STOCHASIC MAP

MANIPULATING QUANTUM REGISTERS

QUANTUM : ALL
- LINEAR MAPS
- MAPPING Q. STATES TO Q. STATES

LINEAR : $\mathbb{O}\left(\alpha|\psi\rangle+\beta|\varphi\rangle\right)=\alpha\,\mathbb{O}(|\psi\rangle)+\beta\,\mathbb{O}(|\varphi\rangle)$

VALID : NORM- PRESERVING

THEOREM : LINEAR + NP $\Rightarrow$ $\mathbb{O}$ IS UNITARY

... REMINDER ... "UNITARY"

$$O|\psi\rangle = |\psi'\rangle$$ [Linear => MATRIX!]

$$O = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1N} \\ \vdots & & \\ \alpha_{N1} & \cdots & \alpha_{NN} \end{pmatrix} \qquad |\psi\rangle = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix}$$

$$|\psi'\rangle = \left[\alpha_{ij}\right]\left(\beta_k\right) = \begin{bmatrix} \vdots \\ \sum_j \alpha_{lj}\beta_j \\ \\ \end{bmatrix} \leftarrow l^{th}$$

$$O_1 O_2 |\psi\rangle = (O_1 O_2)|\psi\rangle \; \ldots$$

Reminder

$$O^{-1}: \quad O O^{-1} = O^{-1} O = \mathbb{1} \equiv \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1N} \\ \vdots & & \\ \alpha_{N1} & \cdots & \alpha_{NN} \end{bmatrix} \qquad U^{\dagger} := \overline{U}^{T} := \begin{bmatrix} \overline{\alpha_{11}} & \cdots & \overline{\alpha_{N1}} \\ \vdots & & \\ \overline{\alpha_{1N}} & & \overline{\alpha_{NN}} \end{bmatrix}$$

UNITARITY: U IS UNITARY IF $U^{\dagger} = U^{-1}$ $\left[U U^{\dagger} = U^{\dagger} U = \mathbb{1} \right]$

Norm-preserving $\rightsquigarrow$ "complex rotations"

SINGLE QUBIT UNITARIES

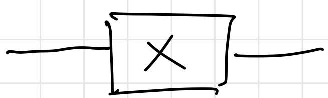
"single qubit quantum gates"

1) IDENTITY ; $U|\psi\rangle = |\psi\rangle$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{1}$$

2) BIT FLIP, NOT, PAULI-X, σ^X, X

$$\left.\begin{aligned} X|0\rangle &= |1\rangle \\ X|1\rangle &= |0\rangle \end{aligned}\right\} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



3) PHASE FLIP, PAULI-Z, σ^z, **Z**

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$-\boxed{Z}-$$

4) HADAMARD GATE H

$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) =: |+\rangle$$
$$H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) =: |-\rangle$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

EVOLUTION (MANIPULATION) & MEASUREMENTS

"C" $= \{|0\rangle, |1\rangle\}$

$|0\rangle - \boxed{Z} -$ "C" $-$

$|1\rangle - \boxed{Z} -$ "C" $-$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

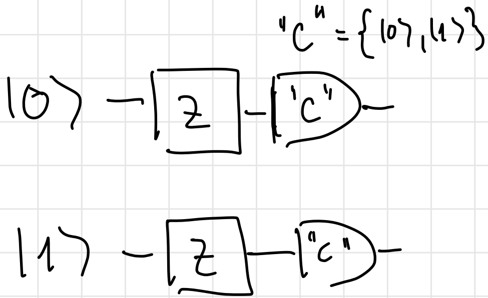
$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$|\alpha|^2$

"0"

"1"

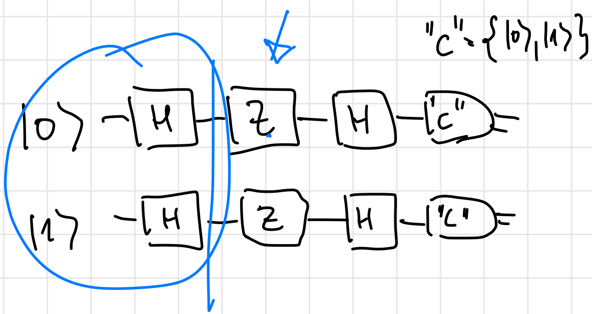
EVOLUTION (MANIPULATION) & MEASUREMENTS



$$\begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}1\\0\end{bmatrix} \searrow$$

$$|\alpha|^2 \nearrow \text{"0"} \qquad |\alpha|^2 \searrow \text{"1"}$$

$$\begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}0\\-1\end{bmatrix} \nearrow$$



WORK IT OUT

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$\underbrace{H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}}$$

$$\underbrace{Z H|0\rangle} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\underline{Z}\,\underline{H}\,\underline{|0\rangle} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$H|0\rangle = |+\rangle =: \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$H|1\rangle = |-\rangle =: \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

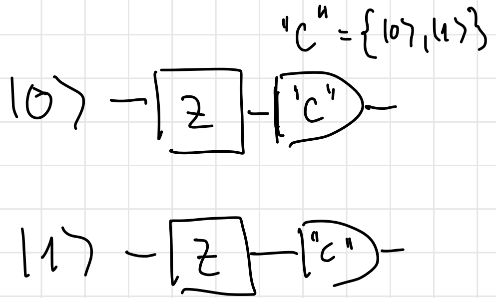
$$H\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = |1\rangle$$

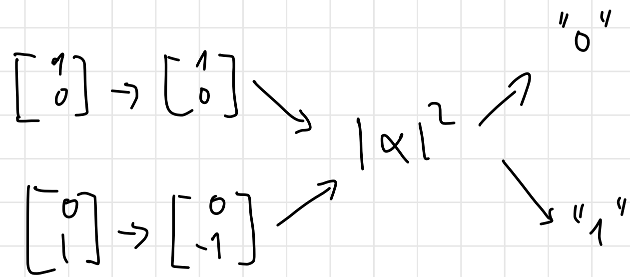
$$H\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}\right] = H\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 1\end{pmatrix}\right] =$$

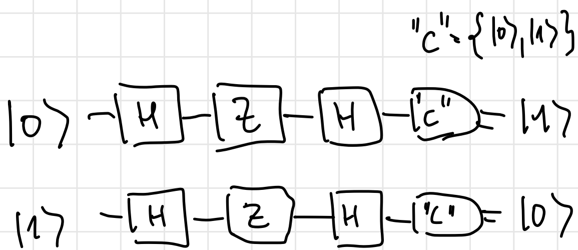
$= |1\rangle$

$$= \frac{1}{\sqrt{2}}H\begin{pmatrix}1\\ 0\end{pmatrix} - \frac{1}{\sqrt{2}}H\begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}0\\ 2\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$$

EVOLUTION (MANIPULATION) & MEASUREMENTS

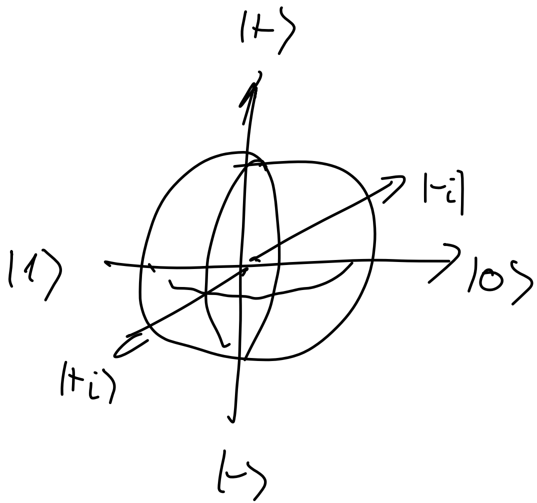


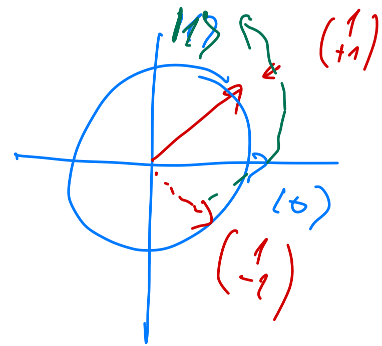




$$\boxed{HZH = X}$$

NOTE: $H = H^{\dagger} \Rightarrow HH = \mathbb{1}$





FOR THE INTERESTED:

"BLOCH SPHERE"

GATES CONTINUED

$\pi/8$ - GATE $\qquad \pi/8 = \begin{bmatrix} 1 & 0 \\ 0 & \exp(i\,\pi/4) \end{bmatrix}$

FIRST LESSON STOPPED HERE

"z" - rotations

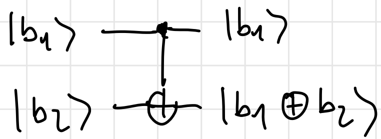
$$R^z(\theta) := R^z_\theta := \begin{bmatrix} 1 & 0 \\ 0 & \exp(i\theta) \end{bmatrix}$$

$$S = \sqrt{Z} \qquad \left[S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \right]$$

$$\pi/8 = \sqrt{S}$$

CORRELATIONS: TWO-QUBIT GATES

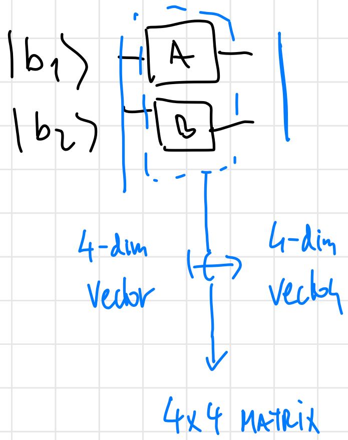
CNOT, CONTROLLED-NOT



CNOT :=  $= \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$$\left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & X \end{array}\right)$$

TWO-QUBIT GATES FROM SINGLE-QUBIT GATES



$$\Longrightarrow \quad |b_1 b_2\rangle = \boxed{A \otimes B} =$$

EXAMPLE

$$\left.\begin{array}{l} |b_1\rangle - \boxed{X} - |1 \oplus b_1\rangle \\ |b_2\rangle - \boxed{Z} - (-1)^{b_2}\,|b_2\rangle \end{array}\right\} \; |1 \oplus b_1\rangle \otimes \left[(-1)^{b_2}|b_2\rangle\right] = (-1)^{b_2}|1 \oplus b_1\rangle |b_2\rangle$$

$b_1 = b_2 = 0 \rightarrow b_1' = 1,\ b_2' = 0$

$$\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \longrightarrow \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

$b_1 = 0$, $b_2 = 1$ $\longrightarrow \hat{b}_1, \hat{b}_2 = 1, 1;$ + "Phase"

$$\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \longrightarrow \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\-1\end{bmatrix}$$

EXAMPLE CONTINUED

$$X \otimes Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 1 \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 1 \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

CNOT is NOT FACTORIZABLE!

$\forall$ single-qubit gates A, B $\left(A, B \in GL_2(\mathbb{C})\right)$

$\mathrm{CNOT} \neq A \otimes B$