Problem 1. (***Spectral decomposition and the bra-notation***)**.**

1. Write down the matrix with respect to the computational basis of $A = \sum_{j\in\{0,1\}^2} \lambda_j |j\rangle \langle j|$.

2. Consider the following Hermitian matrix $B = \frac{1}{2}\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$. Given that $\{|+\rangle, |-\rangle\}$ are the eigenvectors of B, write down the spectral decomposition of B.

3. Using the spectral decomposition of B, write down the spectral decomposition of B^{-1}.

Problem 2. (***Spectral decomposition and the inverse of an Hermitian***)**.**
Let $A \in \mathbb{C}^{N\times N}$ be an Hermitian matrix with eigenvectors $\{|\psi_j\rangle\}_{j=1}^N$ and eigenvalues $\{\lambda_j\}_{j=1}^N$. Let $|b\rangle \in \mathbb{C}^N$ be an arbitrary vector and consider its spectral decomposition w.r.t. A given by

$$|b\rangle = \sum_{j=1}^N \alpha_j |\psi_j\rangle .$$

Show that the following identity holds

$$A^{-1}|b\rangle = \sum_{j=1}^N \alpha_j \lambda_j^{-1} |\psi_j\rangle .$$

Remark. *HHL uses the above identity to solve the QLS problem by applying HS and QPE to estimate the eigenvalues λ_j and afterwards inverting these eigenvalues.*

Problem 3. (***HS and QPE to estimate eigenvalues of Hermitian matrix***)**.**
Let $H \in \mathbb{C}^{N\times N}$ be Hermitian with eigenvectors $\{|\psi_j\rangle\}_{j=1}^N$ and eigenvalues $\{\lambda_j\}_{j=1}^N \subset \mathbb{R}$.

1. What are the eigenvalues of the unitary $e^{iH} = \sum_{j=0}^\infty \frac{(iH)^j}{j!}$?

   Hints.
   - *You may use that $\{|\psi_j\rangle\}_{j=1}^N$ are also the eigenvectors of e^{iH}.*
   - *Moreover, remark that for any $a \in \mathbb{R}$ it holds that $\sum_{j=0}^\infty \frac{(ia)^j}{j!} = e^{ia}$.*

If H is sparse, we can implement the unitary e^{iH} as a quantum circuit using Hamiltonian simulation. Moreover, we can apply the quantum phase estimation routine to it to obtain estimates of the eigenvalues of e^{iH}.

By the previous exercise, there is a clear relation between the eigenvalues of H and those of e^{iH}. Our goal is to obtain estimates of the eigenvalues of H from the eigenvalues of e^{iH}.

2. Suppose λ_1 and λ_2 are two eigenvalues of H such that $\lambda_1 = \lambda_2 + 2k\pi$, for some $k \in \mathbb{Z}$. What can you say about the corresponding eigenphases $\phi_1, \phi_2 \in [0, 2\pi)$ of e^{iH}? Explain why is this a problem when applying QPE to e^{iH} in order to obtain estimates of the eigenvalues of H?

In the previous exercises you note that the mapping from the eigenvalues of H to those of e^{iH} is not one-to-one in general. However, if one knows an upperbound for $\lambda_{max} = \max_j\{\lambda_j\}$, then one can resolve this as follows.

3. Consider the Hermitian matrix $H' = \lambda_{max}^{-1} H$. Show how applying HS and QPE to H' allows one to estimate the eigenvalues of H if you can prepare the states $|\psi_j\rangle$.