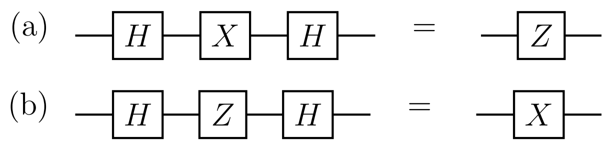
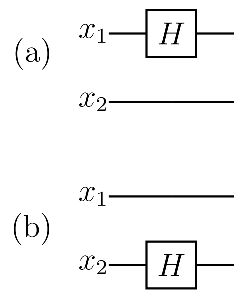
Remark. *The symbols and matrices of the gates discussed in the problems are on page 3.*

Problem 1. (***Circuit identities and matrices [N&C sections 4.2 and 4.3]***).

1. Verify the following circuit identities



2. Give the 4x4 unitary matrix for the following circuits



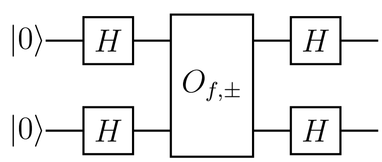
3. Construct a $CNOT$ gate from one controlled-Z gate and two Hadamard gates.

   Remark. *A controlled-Z gate maps $|11\rangle \mapsto -|11\rangle$ and acts as the identity on the other basis states. That is, the matrix of the controlled-Z gate is given by*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Problem 2. (***Modifications of the Deutsch-Jozsa algorithm***).

1. Compute the output of the Deutsch-Jozsa algorithm for $n = 2$ qubits as given below



   Remark.
   - *The phase-oracle $O_{f,\pm}$ maps $|j\rangle \to (-1)^{f(j)}|j\rangle$, where f is a function that maps $f : \{0,1\}^2 \to \{0,1\}$.*
   - *Here we do not yet assume anything about the function f. Your answer will therefore contain expressions such as $(-1)^{f(01)}$ that you cannot simplify any further.*

2. Compute the amplitude of the $|00\rangle$ state in the cases that

   (a) f is constant, i.e., $f(j) = 0$ for all $j \in \{0,1\}^2$ or $f(j) = 1$ for all $j \in \{0,1\}^2$.

(b) f is balanced, i.e., $|f^{-1}(1)| = 2$.

3. Give the entire output state of the above circuit in the above two cases.

4. Show how to use the above circuit to solve the *modified Deutsch-Jozsa problem.*
Modified Deutsch-Jozsa problem.
Input: A phase oracle $O_{f,\pm}$ for a function $f : \{0,1\}^2 \to \{0,1\}$ that satisfies

(a) $f(00) = f(01) = 0$ and $f(10) = f(11) = 1$ or

(b) $f(00) + f(01) + (1 - f(10)) + (1 - f(11)) = 2$.

Output: Decide whether case (a) or (b) holds.

5. (H) Show that the Deutsch-Jozsa algorithm can also distinguish the following cases

(a) f is *approximately constant*, i.e., $|f^{-1}(1)| = 2^n + C$, for a (small) constant C.

(b) f is *approximately balanced*, i.e., $|f^{-1}(1)| = 2^{n-1} + C$, for a (small) constant C.

6. (H) Show how to use the circuit of the Deutsch-Jozsa algorithm to solve the following.
Bernstein-Vazirani problem.
Input: A phase oracle $O_{f,\pm}$ for a function $f : \{0,1\}^n \to \{0,1\}$ that satisfies

$$f(i) = (i \cdot a) \mod 2 = \left(\sum_{k=0}^{n-1} i_k \cdot a_k \right) \mod 2,$$

for some unknown $a \in \{0,1\}^n$.
Output: Find this unknown bitstring $a \in \{0,1\}^n$.

Remark.

- *Convince yourself that the state after the phase-oracle is*

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i \in \{0,1\}^n} (-1)^{i \cdot a} |i\rangle .$$

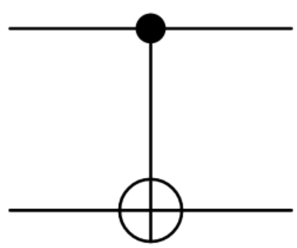
- *Convince yourself of the following equalities*

$$H^{\otimes n} |a\rangle = \otimes_{j=1}^{n} H |a_j\rangle = \otimes_{j=1}^{n} \frac{1}{\sqrt{2}} (|0\rangle + (-1)^{a_j} |1\rangle) = \frac{1}{\sqrt{2^n}} \sum_{i \in \{0,1\}^n} (-1)^{i \cdot a} |i\rangle = |\psi\rangle$$

- *Remember that* $H^{\otimes n} H^{\otimes n} = I$.

Name	Symbol	Matrix
Hadamard	H	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Pauli-X	X	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-Y	Y	$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli-Z	Z	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Phase	S	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
$\pi/8$	T	$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

Figure 4.2. Names, symbols, and unitary matrices for the common single qubit gates.



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The symbol and matrix for the $CNOT$ gate.