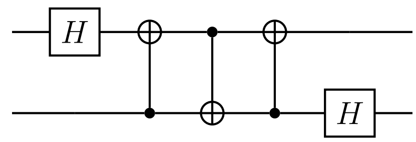
Problem 1. (***Evaluating a quantum circuit***)**.** Consider the following quantum circuit



1. Compute the output state for the initial states $|00\rangle, |01\rangle, |10\rangle$ and $|11\rangle$.
2. Compute the measurement outcome probabilities Pr(00), Pr(01), Pr(10) and Pr(11) for all of the above output states.
3. Compute the matrix that corresponds to the above circuit and use it to compute vector representation of the output state.
4. Redo part 1. & 2. for the circuit with the first H-gate removed.
5. Redo part 1. & 2. for the circuit with both the first and final H-gate removed.
6. Are the matrices of the circuit from part 1. and 5. the same? Explain.

Problem 2. (***Tensor product and Kronckerproduct of vectors***)**.** Consider the states

$$|\psi\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)^{\otimes 2}$$
$$|\phi\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)^{\otimes 3}$$

1. Find the $\alpha_j, \beta_j \in \mathbb{C}$ such that $|\psi\rangle = \sum_{j\in\{0,1\}^2} \alpha_j |j\rangle$ and $|\phi\rangle = \sum_{j\in\{0,1\}^3} \beta_j |j\rangle$.
2. Compute the vector representation of both $|\psi\rangle$ and $|\phi\rangle$ using the Kronecker product.

Problem 3. (***Rewriting a state in the Hadamard basis***)**.** Rewrite the 2-qubit state

$$|\psi\rangle = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

in the Hadamard basis.

Remark. *For $c \in \{+,-\}^2$ you can use that $\langle c \mid \psi\rangle = \frac{1}{2}\langle c \mid 00\rangle - \frac{1}{2}\langle c \mid 01\rangle + \frac{1}{\sqrt{2}}\langle c \mid 11\rangle$.*

Problem 4. (*Unitary matrices*). Consider the Hadamard gate given by

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

1. Verify that H is a unitary matrix.

2.* Let U be an $n \times n$ complex matrix. Prove the following statement

$$U \text{ is norm-preserving}^1 \implies U \text{ is invertible.}$$

(You are not allowed to use that unitary matrices satisfy $U^*U = I$.)
Hint 1: If U is norm-preserving, what vectors will be mapped to the all zero vector?
Hint 2: You are allowed to use that $|| \, |v\rangle \, || = 0 \implies |v\rangle = 0$.
Hint 3: Remember that $\ker(U) = \{0\} \implies U$ is invertible.

Remark. *For more exercises, see Chapter 1 (exercises 1-5) and Chapter 2 (exercises 1, 3, & 4) of Ronald de Wolf's lecture notes (`https://homepages.cwi.nl/~rdewolf/qcnotes.pdf`) and/or the relevant chapters of Nielsen's & Chuang's "Quantum Computation and Quantum Information".*

[1] $||U\,|v\rangle\,|| = ||\,|v\rangle\,||$ for all $|v\rangle \in \mathbb{C}^n$