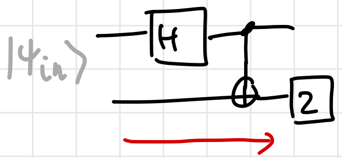
Solutions to

Take Home Assignments

PROBLEM 1.1.



(a) Matrix: $(\mathbb{1}\otimes Z)\,\mathrm{CNOT}\,(H\otimes\mathbb{1})\,|\psi_{in}\rangle$ NOTE ORDER.

In block form:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & X \end{pmatrix}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} Z & 0 \\ 0 & ZX \end{pmatrix}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} Z & Z \\ ZX & -ZX \end{pmatrix}$$

$$ZX = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(b) $U|01\rangle = U\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 2^{nd}$ column of $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

(c) $|\langle 00|U|10\rangle|^2 = \frac{1}{2}\left|(1\,0\,0\,0)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right|^2 = \frac{1}{2}$

PROBLEM 1.2.

(a) H is hermitian & unitary $\Rightarrow$ eigenvalues must be $+1, -1$

Proof: $\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1+\sqrt{2}\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}2+\sqrt{2}\\\sqrt{2}\end{pmatrix} = \begin{pmatrix}\sqrt{2}+1\\1\end{pmatrix}$

$\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1-\sqrt{2}\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}2-\sqrt{2}\\-\sqrt{2}\end{pmatrix} = -\begin{pmatrix}1-\sqrt{2}\\1\end{pmatrix}$

Eigenvectors of H = $|v_0\rangle = \begin{pmatrix}1+\sqrt{2}\\1\end{pmatrix}$, $|v_1\rangle = \begin{pmatrix}1-\sqrt{2}\\1\end{pmatrix}$

Z is diagonal, ev-ev pairs are $|u_0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$; 1 ; $|u_1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$; -1.

$\hookrightarrow$ diag $\Rightarrow$ evecs are canonical (comp. basis)

$\Rightarrow$ eval's are on the diagonal

By bilinearity & mixed-product property

Evecs of $H \otimes Z$ are $\{|v_1\rangle\, |v_2\rangle\} \otimes \{|u_1\rangle, |u_2\rangle\}$ (pairwise)

Evals are corresponding products

So. for $b_1, b_2 \in \{0,1\}$, the $(b_1 b_2)_2$ eigenvector

of $H \otimes Z$ is $|\psi_{b_1 b_2}\rangle = |v_{b_1}\rangle \otimes |u_{b_2}\rangle$

with eigenvales $(-1)^{b_1} \cdot (-1)^{b_2}$

The evs of $CNOT\,(H \otimes Z)\,CNOT$ are then

$CNOT\,|\psi_{b_1 b_2}\rangle$; conjugation preserves eigenvalues so:

$b_1, b_2 =$ 0,0 ; 0,1 ; 10 ; 11

$\begin{pmatrix}1+\sqrt{2}\\0\\0\\1\end{pmatrix}, 1$; $\begin{pmatrix}0\\1+\sqrt{2}\\1\\0\end{pmatrix}, -1$; $\begin{pmatrix}1-\sqrt{2}\\0\\0\\1\end{pmatrix}, -1$; $\begin{pmatrix}0\\1-\sqrt{2}\\1\\0\end{pmatrix}, 1$

PROBLEM 1.2 continued

(b) $U = H Z_{\pi/2} H Z$; Note $HZH = X$.

I conjugate U by H. $U \to HUH =$

$$= Z_{\pi/2} X = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}$$

e-values: $-\frac{1+I}{\sqrt{2}} = -\sqrt{I}$ $\qquad \sqrt{I} = \frac{1+I}{\sqrt{2}}$

NB:

$$(-1)^{3/4} = -\sqrt{-I}$$

e-vectors: $\tilde{V}_0 = \begin{pmatrix} -\frac{1-I}{\sqrt{2}} \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} -(-1)^{3/4} \\ 1 \end{pmatrix} = \tilde{V}_1$

$$\begin{pmatrix} -\sqrt{-I} \\ 1 \end{pmatrix} \qquad \begin{pmatrix} \sqrt{-I} \\ 1 \end{pmatrix}$$

$$V_0 = H\tilde{V}_0 = \begin{pmatrix} 1-\sqrt{-I} \\ -\sqrt{-I}-1 \end{pmatrix} = \begin{pmatrix} I - I\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} i(1-\sqrt{2}) \\ 1 \end{pmatrix}$$

$$V_1 = H\tilde{V}_1 = \begin{pmatrix} \frac{1}{2}\left(\sqrt{2}+(1-i)\right) \\ -\frac{1}{\sqrt{2}}\left(1-\sqrt{-I}\right) \end{pmatrix} = \begin{pmatrix} I + I\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} i(1+\sqrt{2}) \\ 1 \end{pmatrix}$$

NB: many different-looking expressions are equal

$$\begin{pmatrix} i(1-\sqrt{2}) \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}(1+I)\begin{pmatrix} 1 \\ I-\sqrt{2}I \end{pmatrix}$$

$$\begin{pmatrix} i(1+\sqrt{2}) \\ 1 \end{pmatrix} = -\frac{\sqrt{2}}{2}(1+I)\begin{pmatrix} 1 \\ I+\sqrt{2}I \end{pmatrix}$$

PROBLEM 1.3.

(a) $\sqrt{\langle a|a\rangle} = 1$; $\sqrt{\langle b|b\rangle} = 1$ $\quad \langle a|b\rangle = \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{3} = 0$.

(b) $|\langle b|H\, Z_{\pi/2}\, H|0\rangle|^2 =$

$$\to |\langle +|S|+\rangle|^2 = \frac{1}{2}$$

$$\to |\langle -|S|+\rangle|^2 = \frac{1}{2}$$

$$|\langle +|S|+\rangle|^2 = \left|\frac{1}{2}(1+i)\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|\langle -|S|+\rangle|^2 = \left|\frac{1}{2}(1-i)\right|^2 \quad \cdots \quad = \frac{1}{2}$$

(C)

$$\left|\langle a|\left(\frac{(1+i)}{2}|0\rangle + \frac{(1-i)}{2}|1\rangle\right)\right|^2$$

$$= \left|\frac{1}{2\sqrt{3}}\left(1+\sqrt{2} + (1-\sqrt{2})i\right)\right|^2$$

$$= \frac{1}{12}\left[2(1+\sqrt{2})^2\right] = \frac{1}{2}.$$

Since $|a\rangle$ & $|b\rangle$ are the complete basis the other inner product must be $\frac{1}{2}$ as well.

PROBLEM 2

Full credit was given for any solution, only mistakes penalized

A more general statement proven below FYI:

(a) A general statement (not required for credit)

Let $|\varphi\rangle$, $|\psi\rangle$ be any two n-qubit states

and let $\{V_1^1 .. V_{j_1}^1\}, \ldots \{V_1^m, \ldots V_{j_m}^m\}$

be m sets of unitaries s.t. $\forall\ 1 \le k \le m$

$V_1^k \otimes .. \otimes V_{j_k}^k$ acts on n qubits,

and let $\{\theta_1^1 .. \theta_{j_1}^1\}, \ldots \{\theta_1^m \ldots \theta_{j_m}^m\}$

$\theta_j^\ell \in [0, 2\pi]$ be (arbitrary) phases.

Then $$\left|\langle\varphi|\left[\bigotimes_{k=1}^{j_m} e^{i\theta_k^m} V_k^m\right]\cdots\left[\bigotimes_{k=1}^{j_1} e^{i\theta_k^1} V_k^1\right]|\psi\rangle\right|$$

$$= \left|\underbrace{e^{i\sum_{\ell=1}^{m}\sum_{k=1}^{j_\ell}\theta_k^\ell}}\langle\varphi|\left[\bigotimes_{k=1}^{j_m} V_k^m\right]\cdots\left[\bigotimes_{k=1}^{j_1} V_k^1\right]|\psi\rangle\right|$$

$$= \left|\langle\varphi|\left[\bigotimes_{k=1}^{j_m} V_k^m\right]\cdots\left[\bigotimes_{k=1}^{j_1} V_k^1\right]|\psi\rangle\right|$$

$\to |e^{i\varphi}\alpha| = |e^{i\varphi}||\alpha| = |\alpha|, \forall \varphi \ldots$

(b) Full credit was given for any solution, only mistakes penalized
A more general statement proven below FYI:

GENERIC RESULT: DETECTABILITY OF LOCAL PHASE, WITH CONSTRUCTION (~~not~~ needed for take-home assignment)

Let $W = \mathbb{I} \oplus V = \begin{bmatrix} \mathbb{1} & \\ & V \end{bmatrix}$

and $U = A\, W\, B$, A, W, B unitaries

let $|v\rangle$ an $e^{i\varphi}$-eigenstate of V.

Then $W \frac{1}{\sqrt{2}}(|0\rangle|v\rangle + |1\rangle|v\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi}|1\rangle)\, |v\rangle$.

Let $W^{\theta} = \mathbb{I} \oplus e^{i\theta} V$

$$W^{\theta} \frac{1}{\sqrt{2}} \left(|0\rangle|v\rangle + |1\rangle|v\rangle\right) = \frac{1}{\sqrt{2}} \left(|0\rangle + e^{i\varphi+\theta}|1\rangle\right)|v\rangle$$

assume $\theta \neq \pi$.

now note $\left|\langle +|\left(|0\rangle + e^{i\varphi}|1\rangle\right)\right| \neq \left|\langle +|\left(|0\rangle + e^{i\varphi+\theta}|1\rangle\right)\right|$

$\forall\, \varphi$, $\forall\, \theta \neq \pi$.

Then measuring with respect any basis which includes the vec. $A^{\dagger}|+\rangle|v\rangle$, starting from $B^{\dagger}|+\rangle|v\rangle$

detects (influences outcome probabilities) between

AWB & $AW^{\theta}B$.

if $\theta = \pi/2$, pick $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\varphi}|1\rangle\right)$.

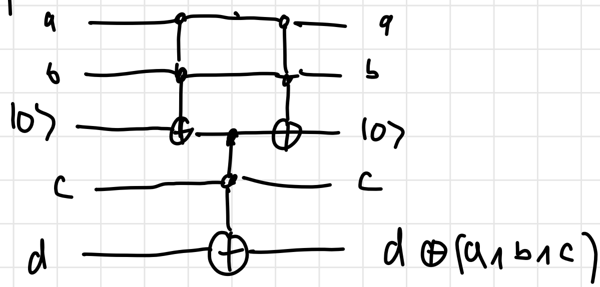
in place of $|+\rangle$.

The objective of the task was to

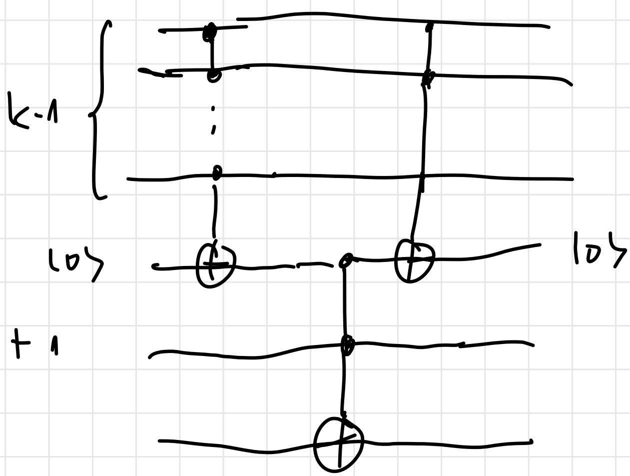
- further evaluate capacity to evaluate circuits
- make you think about how phases multiplying unitaries percolate into amplitudes of q. states
- make you note how measurement bases affect what aspects of amplitudes can be detected.
- develop a feeling for the "overcompleteness" of basic linear algebra description of QM.

PROBLEM 3

(a)



(b) naively from k-1 to k) k-2 ancillas will do.

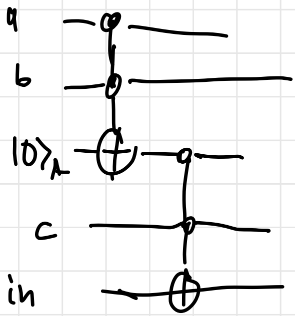


... induction

Much more space-efficient schemes possible utilizing Phases (think: Toffoli circuit does a similar thing.)

IMPORTANT

WHY is:



(construction 1)

not a valid solution?

Answer: THIS DOES NOT IMPLEMENT <u>the unitary</u>

$\mathbb{1}_{14} \oplus X$ =  on wires a, b, c.

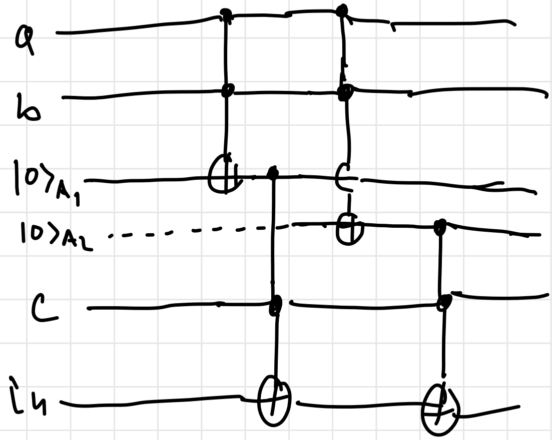
How to see this,

1.) Set $|+\rangle_a |1\rangle_b |1\rangle_c |0\rangle_{in}$ on input.

After evaluation, observe <u>ancilla</u>:

it is entangled to a. If you trow it away, superposition is broken. OR:

2) ctrl-Toffoli is SELF-inverse. The construction I suggests a new ancilla per call (it is set to $|0\rangle$)



This is not $\mathbb{1}$ on a, b, c, in, and it must be set $|+\rangle_a |1\rangle_b |1\rangle_c |0\rangle_{in}$ to see...

But then the construction is Not ctr-Toffoli

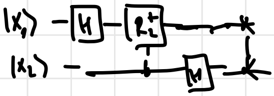
... not a "dirty trick".. you were advised to "see page 183 of N & C"...

LESSON: GARBAGE COLLECTION (uncomputing ancilla qubits) is necessary for Q.C.

PROBLEM 4

1.) $\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$

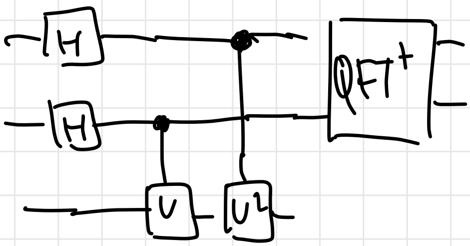
$R_2^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{\pi}{2} i} \end{pmatrix}$



Notice, QFT is symmetric in comp basis
so $QFT^\dagger = QFT^*$, so i can just conjugate R_2!
(inversing circuit and conjugating all is also ok.)

2.) Computed before

3)



4) $|-\rangle$ is an $e^{i\frac{\pi}{2}}$-eigenvector of U!

$i = \exp\left(\frac{\pi}{2} i\right) = \exp\left(0.\underbrace{01}\ 2\pi i\right)$

$\Rightarrow$ By correctness of QPE, output is $|0\rangle|1\rangle|-\rangle =$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} / \sqrt{2}$$

NB: it was only asked to produce the <u>output state</u> in vector form...

5)

$$|0\rangle|0\rangle|-\rangle \rightarrow |+\rangle|+\rangle|-\rangle \rightarrow$$

$$\frac{1}{\sqrt{2}}|+\rangle\left(|0\rangle + i|1\rangle\right)|-\rangle \rightarrow$$

$$\frac{1}{2}\left(|0\rangle - |1\rangle\right)\left(|0\rangle + i|1\rangle\right)|-\rangle \rightarrow \text{QFT}^\dagger \text{ property}$$

$$\rightarrow |0\rangle|1\rangle|-\rangle$$

PROBLEM 5.1

FOR PART (a) THE POINT WAS

a) TO OBSERVE That Grover can be applied,
i.e. that the "database" is the list of integers up to w,
and to recall that given k "marked elements"
the complexity is $\sqrt{\frac{w}{k}}$;

b) to show that for w-smooth SQUARE FREE numbers
(N is square free if all prime factors occur at most once)
the number of divisors below w must be $\log_w N$...
Since we forgot to specify square-free-ness,
full credit was given for a) alone

Proof. Let $N = \prod_{j=1}^{k} p_j$, $p_j \neq p_\ell \ \forall j \neq \ell$, p_j prime.

Claim: $\bullet \ k \geq \lfloor \log_w N \rfloor$

Proof. $k < \lfloor \log_w N \rfloor \Rightarrow N < w^{\lfloor \log_w N \rfloor} \leq w^{\log_w N} = N$

so $N < N$ ↯.

Note $\log_w N = \frac{\log N}{\log w}$. Also note $p_j \leq w$, so

$$\Rightarrow O\left(\sqrt{\frac{w}{k}}\right) = O\left(\sqrt{\frac{w \log w}{\log CN}}\right)$$

PROBLEM 5.1.

$$\frac{1}{\sqrt{2^n}}\sum |i\rangle \to \sum f(i)|i\rangle = \frac{1}{\sqrt{2^n}}\sum (-1)^{\overbrace{(i)_2\cdot(s_2) \bmod 2}^{\text{per def. of } f.}} |i\rangle$$

The claim is that for $|\psi\rangle = \frac{1}{\sqrt{2^n}}\sum (-1)^{(i)_2\cdot(s_2) \bmod 2}|i\rangle$.

$$|\langle s|H^{\otimes n}|\psi\rangle|^2 = 1.$$

Note $H = H^{+}$. It suffices to prove $H^{\otimes n}|s\rangle = |\psi\rangle$.

$$H|s\rangle = \bigotimes X^{s_i}|+\rangle, \quad X^{s_i} = \begin{cases} 1\!\!1 & s_i = 0 \\ X & s_i = 1 \end{cases}$$

$$\propto \bigotimes_i \left(|0\rangle + (-1)^{s_i}|1\rangle\right)$$

$$= \sum_{j_1, j_2 \dots j_n} (-1)^{j_1\cdot s_1}(-1)^{j_2\cdot s_2}\dots(-1)^{j_n\cdot s_n}|j_1\rangle \dots |j_n\rangle \quad (*)$$

why? because
(-1) appears only if both $j_k = 1$
AND $s_k = 1$
"x" is boolean AND...

note $(-1)^{\sum j_k\cdot s_k} = (-1)^{\sum j_k\cdot s_k \bmod 2} \Rightarrow \; = \sum_j (-1)^{\vec{j}\cdot\vec{s}}|\vec{j}\rangle$ as claimed ∎

Note (*) is the only non-triv observation...