ENS DE LYON

TD de Sémantique et Vérification

IX– Bisimulations and Filter Models

Tuesday 2nd April 2019

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In this set of exercises, we will discuss bisimulations and their relation to LTL, and properties of ultrafilters and filter models.

Bisimulations and Logical Equivalence

We first recall bisimulations and bisimilarity. Let $TS_i = (S_i, \mathrm{Act}, \to_i, I_i, \mathrm{AP}, L_i)$ for $i = 1, 2$ be transition systems with the same set of actions and atomic propositions. We say that $R \subseteq S_1 \times S_2$ is a *bisimulation* if for all $x \in S_1$ and $y \in S_2$ with $x \mathrel{R} y$, we have

- $L_1(x) = L_2(y)$;
- for all $a \in \mathrm{Act}$, if $x \xrightarrow{a}_1 x'$, then there is y' with $y \xrightarrow{a}_2 y'$ and $x' \mathrel{R} y'$; and
- for all $a \in \mathrm{Act}$, if $y \xrightarrow{a}_2 y'$, then there is x' with $x \xrightarrow{a}_1 x'$ and $x' \mathrel{R} y'$.

We say that $x \in S_1$ and $y \in S_2$ are *bisimilar*, written $x \sim y$, if there is a bisimulation R with $x \mathrel{R} y$.

Given TS_1 and TS_2 as above, we say that $s_1 \in S_1$ and $s_2 \in S_2$ are *Hennessy-Milner-equivalent*, written $s_1 \equiv_{\mathrm{HM}} s_2$, if for all Hennessy-Milner formulas φ, we have that $s_1 \models \varphi \iff s_2 \models \varphi$.

Exercise 1.

1. Let I be a non-empty set and R_i bisimulations (between TS_1 and TS_2) for all $i \in I$. Show that $\bigcup_{i \in I} R_i$ is a bisimulation.
2. Conclude that $\sim = \bigcup\{R \mid R \text{ is bisimulation}\}$.
3. Let $\mathrm{Rel} = \{R \mid R \subseteq S_1 \times S_2\}$ be the poset of relations between S_1 and S_2. Define an $f : \mathrm{Rel} \to \mathrm{Rel}$, such that R is a bisimulation if and only if $R \subseteq f(R)$. Show that $\sim$ is the greatest fixed point of f.

Exercise 2.

Let TS_1 and TS_2 be transition systems over the same set of atomic propositions AP and set of action Act and $s_i \in S_i$ with $s_1 \sim s_2$. Show that $s_1 \equiv_{\mathrm{HM}} s_2$.

Exercise 3.

Show that $\sim \subseteq S \times S$ is an equivalence relation on a single transition system TS with states S.

Ultrafilters

First of all, we recall some terminology. Let S be a non-empty set. We say that $F \subseteq \mathcal{P}(S)$ is a *filter* on S, if

- $\emptyset \notin F$;
- for all $A, B \in F$ there is a $C \in F$ with $C \subseteq A$ and $C \subseteq B$ (downwards directed); and
- for all $A \in F$ and $B \subseteq S$, if $A \subseteq B$ then $B \in F$ (upwards closed).

A filter F is called

- *ultrafilter*, if for all $A \subseteq S$, either $A \in F$ or $A^{\complement} \in F$; and
- *maximal*, if for all filters G with $F \subseteq G$, we have $F = G$.

Recall also the finite intersection property (FIP): A set $U \subseteq \mathcal{P}(S)$ is said to have the finite intersection property, if for every finite set $V \subseteq U$, $\bigcup V \in U$.

Finally, we will need Zorn's lemma. Let $(P, \leq)$ be a partially ordered set. We call $C \subseteq P$ a *chain*, if for all $x, y \in C$, we either have $x \leq y$ or $y \leq x$. Moreover, an element x of P is called *maximal*, if $y \leq x$ for all $y \in P$. Zorn's lemma asserts now that if every chain in P has an upper bound, then P has a maximal element.

Exercise 4.
Show that a filter is an ultrafilter if and only if it is maximal.

Exercise 5.
Let S be a non-empty set.

1. Suppose that $U \subseteq \mathcal{P}(S)$ has the FIP. Let
$$F = \left\{ A \subseteq S \;\middle|\; \exists n \in \mathbb{N}.\, \exists A_1, \ldots, A_n \in S.\, \bigcap_{i=1}^{n} A_i \subseteq A \right\}.$$
Show that F is a filter over S with $S \subseteq F$.
2. Show that every filter F over S can be extended to a maximal filter by using Zorn's lemma.
3. Conclude that every $U \subseteq \mathcal{P}(S)$ that has the FIP can be extended to an ultrafilter.

Final Remark: In the above definitions and results about (ultra)filters, the poset $(\mathcal{P}(S), \subseteq)$ can be replaced by any, so-called, Boolean algebra. This is, however, not relevant to the results of the lecture and is left for your own further studies.