ENS DE LYON

TD de Sémantique et Vérification

IV– Topological Aspects of Linear Time Properties

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Henning Basold
henning.basold@ens-lyon.fr

In this set of exercises, we will discuss topological characterisations of safety and liveness properties.

Topological Spaces

- A *topological space* is a pair $(X, \mathcal{U})$ of a set X and a subset $\mathcal{U}$ of $\mathcal{P}(X)$, called the *open sets of* X, such that
  1. $\emptyset \in \mathcal{U}$ and $X \in \mathcal{U}$;
  2. for any set I and family $\{U_i \in \mathcal{U}\}_{i \in I}$, also $\bigcup_{i \in I} U_i \in \mathcal{U}$; and
  3. for all $U, V \in \mathcal{U}$, also $U \cap V \in \mathcal{U}$.

  A set $U \in \mathcal{U}$ is called *open* and elements $x \in X$ are called points. If $\mathcal{U}$ is clear from the context, we often refer to X is the topological space.
- Given a point $x \in X$, we say that N is a *neighbourhood* of x if there is an open set U, such that $x \in U$ and $U \subseteq N$. The collection of all neighbourhoods of x is denoted by $\mathcal{N}_x$.
- Given a topological X, we say that $F \in X$ is *closed*, if $X \setminus F$ is open.

Exercise 1.
Show that
1. $\emptyset$ and X are closed;
2. for any set I and family $\{F_i \text{ closed}\}_{i \in I}$, also $\bigcap_{i \in I} F_i$ closed; and
3. for all closed F and G, also $U \cup V$ is closed.

For any set $S \subset X$, we define the *closure* $\overline{S}$ of S by

$$\overline{S} = \bigcap \{F \subseteq X \mid F \text{ closed and } S \subseteq F\},$$

which makes sense by the previous exercise.

Exercise 2.
1. Show that $\overline{S} = \{x \in X \mid \forall N \in \mathcal{N}_x.\, N \cap S \neq \emptyset\}$.
2. Show that S is closed iff $\overline{S} = S$.

Metric Spaces

- A *metric space* is a pair (X, d), where X is a set and d is a map $d \colon X \times X \to \mathbb{R}_{\geq 0}$, such that for all $x, y, z \in X$
  1. $d(x, y) = 0$ iff $x = y$ (positive definiteness);
  2. $d(x, y) = d(y, x)$ (symmetry); and
  3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

  The map d is then called a *metric*.
- Given a metric space (X, d), $x \in X$ and $\varepsilon > 0$, we define the *ε-ball* $B_\varepsilon(x)$ around x by

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

Exercise 3.

Let (X, d) be a metric space and define $\mathcal{U} \subseteq \mathcal{P}(X)$ by

$$\mathcal{U} = \{U \subseteq X \mid \forall x \in U. \exists \varepsilon > 0. B_\varepsilon(x) \subseteq U\}.$$

1. Show that the thus defined $(X, \mathcal{U})$ is a topological space.
2. Show that for any $S \subseteq X$ that we have $\overline{S} = \{x \in X \mid \forall \varepsilon > 0. B_\varepsilon \cap S \neq \emptyset\}$.

Let Σ be a finite set, called an alphabet. Previously, Σ was given by $\mathcal{P}(\mathrm{AP})$. However, now the internal structure of Σ is not relevant, which is why we work with an arbitrary alphabet. The set of infinite sequences over Σ is denoted by Σ^ω as before. Let $d \colon \Sigma^\omega \times \Sigma^\omega \to \mathbb{R}_{\geq 0}$ be given by

$$d(\sigma, \tau) = \begin{cases} 0, & \sigma = \tau \\ 2^{-\min\{k \in \mathbb{N} \mid \sigma(k) \neq \tau(k)\}}, & \sigma \neq \tau \end{cases}$$

Let us also denote by $\sigma|_n$ the prefix of length n of σ.

Exercise 4.

1. Show that (Σ^ω, d) is a metric space. (*Hint: It will be beneficial here and later to establish a different characterisation of $d(\sigma, \tau) < 2^{-n}$ in terms of prefixes of length n for $n \in \mathbb{N}$.*)
2. Show that the closed sets of Σ^ω are exactly the safety properties. (*Hint: Use the characterisation of safety properties from the previous exercise and the above characterisation of closed sets. You can also assume, without loss of generality, that any given $\varepsilon > 0$ is of the form $\frac{1}{2^n}$.*)
3. A set $D \subseteq X$ in a topological space X is called *dense*, if $\overline{D} = X$. Show that the dense subsets of Σ^ω are exactly the liveness properties.

Limits and Cauchy Sequences

Given a metric space (X, d), we say that a sequence $(x_n)_{n=0}^\infty$ in X with $x_n \in X$ *converges to* $x \in X$, if

$$\forall \varepsilon > 0. \exists N \in \mathbb{N}. \forall n \geq N. d(x_n, x) < \varepsilon.$$

We say that x is the *limit* of $(x_n)_{n=0}^\infty$ and write $x = \lim_{n\to\infty} x_n$. It is easily verified that limits are unique, and that $x = \lim_{n\to\infty} x_n$ iff $\forall N \in \mathcal{N}_x. \exists N \in \mathbb{N}. \forall n \geq N. x_n \in N$.

A special type of sequences are Cauchy sequences. We call a sequence $(x_n)_{n=0}^\infty$ in X a *Cauchy sequence*, if

$$\forall \varepsilon > 0. \exists N \in \mathbb{N}. \forall n, m \geq N. d(x_n, x_m) < \varepsilon.$$

The metric space X is called *complete*, if every Cauchy sequence in X converges.

Exercise 5.

1. Let (X, d) be a metric space and $S \subseteq X$ Show that $\overline{S}$ consists of all those points that are the limit of a sequence in S.
2. Show that the space (Σ^ω, d) is Cauchy complete.