Enriched and Homotopical Coalgebra

Topology, Algebra, and Categories in Logic 2024, Barcelona

Henning Basold

Leiden Institute of Advanced Computer Science

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Universiteit Leiden
The Netherlands

Outline

Introduction and Motivation

Motivation

Enrichment in Coalgebra and Modal Logic

- Semantics of effectful computations and free algebras (Kleisli)
- Modal logic that respects some structure[1] (order, topology)

Homotopy theory and algebraic topology for behaviour

- Concurrent computing — detecting deadlocks[2]
- Distributed computing — computability results[3]
- Hybrid computing — detecting and handling Zeno behaviour[4]
- Homotopy-invariant modal logic for higher dimensional automata[5]

[1] Yde Venema, Jim de Groot, and Nick Bezhanishvili (Dec. 8, 2022). "Coalgebraic Geometric Logic: Basic Theory". In: *Logical Methods in Computer Science* Volume 18, Issue 4; Adriana Balan, Alexander Kurz, and Jiří Velebil (Sept. 22, 2015). "Positive Fragments of Coalgebraic Logics". In: *Logical Methods in Computer Science* Volume 11, Issue 3.

[2] Lisbeth Fajstrup et al. (2016). *Directed Algebraic Topology and Concurrency*. Springer.

[3] Maurice Herlihy, Dmitry Kozlov, and Sergio Rajsbaum (2013). *Distributed Computing Through Combinatorial Topology*. 1st ed. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc.

[4] Aaron D. Ames and Shankar Sastry (June 2005). "Characterization of Zeno Behavior in Hybrid Systems Using Homological Methods". In: *Proceedings of the 2005, American Control Conference, 2005.* ACC 2005.

[5] Cristian Prisacariu (2014). *Higher Dimensional Modal Logic*. arXiv: 1405.4100. URL: http://arxiv.org/abs/1405.4100. preprint.

Enriched Kleisli Categories

Goal

- Kleisli category of a monad gives effecful semantics or free algebras
- Often want extra structure on Kleisli morphisms coming from monad, like order, CPO, topology, metric etc.
- Some specific results in literature
- Here: general enrichment from structure on functor
- Example: powerset gives order-enrichment on morphisms $X \to \mathcal{P}(Y)$

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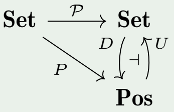
- Example: powerset gives order-enrichment on morphisms $X \to \mathcal{P}(Y)$

Example (Powerset monad)



$\mathcal{P} = U \circ P$, $\mathbf{Pos}$ nicely enriched over $\mathbf{Set}$

Nice Enrichment

Definition

- $(\mathcal{V}, \times, *)$ a closed symmetric monoidal category
- $(\mathcal{M}, \odot, 1)$ closed symmetric $\mathcal{V}$-monoidal $\mathcal{V}$-category, $\underline{\mathcal{M}}_{\mathcal{M}}$ self-enriched

$\mathcal{M}$ is closed over $\mathcal{V}$, if there is a $\mathcal{V}$-adjunction $L : \mathcal{V} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{M} : U$ with L and U strong monoidal and $U\underline{\mathcal{M}}_{\mathcal{M}}(m, n) \cong \underline{\mathcal{M}}(m, n)$ in $\mathcal{V}$.

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Example

- $(\mathbf{Set}, \times, *)$ sets with Cartesian product
- $(\mathbf{Pos}, \times, *)$ posets with Cartesian product and $\mathbf{Set}$-enriched
- $\underline{\mathbf{Pos}}$ is $\mathbf{Pos}$-enriched with point-wise order on monotone maps
- $D : \mathbf{Set} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Pos} : U$ — discrete order-forgetful adjunction
- Forgetting the order yields: $U\underline{\mathbf{Pos}}(P, Q) = \mathbf{Pos}(P, Q)$

Tensor and Cotensor from Adjunction

Lemma

If $(\mathcal{M}, \odot, 1)$ *closed over* $(\mathcal{V}, \times, *)$ *with* $L : \mathcal{V} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{M} : U$*, then* $\mathcal{M}$ *is*

- *tensored over* $\mathcal{V}$*:* $v \otimes m = Lv \odot m$
- *cotensored over* $\mathcal{V}$*:* $\{v, m\} = \underline{\mathcal{M}}_{\mathcal{M}}(Lv, m)$

Example (Powerset monad)

- $X \otimes (Q, \leq) = (X \times Q, \sqsubseteq_{\text{discr}} \times \leq)$
- $\{X, (Q, \leq)\} = ([X, Q], \leq_{\text{pw}})$ (point-wise order on maps)

Getting Enrichment for Kleisli

Definition

- $(\mathcal{M}, \odot, 1)$ be closed over $(\mathcal{V}, \times, *)$ with $L : \mathcal{V} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{M} : U$
- A monad $\mathbb{T}$ presented by Kleisli triple (T, η, e) on $\mathcal{V}$ is Kleisli $\mathcal{M}$-enriching, if there is a $\mathcal{V}$-functor $R\colon \mathcal{V} \to \mathcal{M}$ with $UR = T$ and for all $v, w \in \mathcal{V}$ a morphism $\tilde{e}_{v,w} \colon \{v, Rw\} \to \underline{\mathcal{M}}_{\mathcal{M}}(Rv, Rw)$ in $\mathcal{M}_0$ such that some diagrams commute.

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Intuition

The morphism $\tilde{e}$ gives an enriched version of Kleisli extension

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Example (Powerset Monad)

- $P \colon \mathbf{Set} \to \mathbf{Pos}$ with $PX = (\mathcal{P}(X), \subseteq)$
- $\tilde{e}_{X,Y} \colon \{X, (\mathcal{P}(Y), \subseteq)\} \to [(\mathcal{P}(X), \subseteq), (\mathcal{P}(Y), \subseteq)]$

$$\tilde{e}_{X,Y}(f)(U) = \bigcup_{x \in U} f(x)$$

Enriching Kleisli Categories

Theorem

For any Kleisli $\mathcal{M}$-enriching monad $\mathbb{T}$, there is an $\mathcal{M}$-category $\underline{\mathbf{Kl}(\mathbb{T})}$ with $U_*(\underline{\mathbf{Kl}(\mathbb{T})}) \cong \mathbf{Kl}(\mathbb{T})$ in $\mathcal{V}$-$\mathbf{Cat}$, where $U_* \colon \mathcal{M}$-$\mathbf{Cat} \to \mathcal{V}$-$\mathbf{Cat}$ is change of enrichment along $U \colon \mathcal{M} \to \mathcal{V}$.

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Example (Powerset Monad)

- $\underline{\mathbf{Kl}(\mathcal{P})}$ is $\mathbf{Pos}$-enriched
- Forgetting the order yields $\mathbf{Kl}(\mathcal{P})$
- Works also for lattice/CPO-enrichment

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Example (Finite Probability Distributions)

- $\mathcal{D} \colon \mathbf{Set} \to \mathbf{Set}$ factors through $\mathbf{Set} \to \mathbf{Top}$
- Discrete-forgetful adjunction $\mathbf{Set} \leftrightarrow \mathbf{Top}$
- Yields $\mathbf{Top}$-enrichment of $\mathbf{Kl}(\mathcal{D})$

Enriched and Monoidal Coalgebra

Formal Coalgebra in 2-Categories

- Work in 2-category $\mathcal{C}$: $\mathbf{Cat}$, $\mathcal{V}$-$\mathbf{Cat}$, $\mathbf{Fib}$, $\mathbf{qCat_2}$ (homotopy 2-category of quasi-categories)[6], $\mathfrak{h}\mathcal{K}$ (homotopy 2-category of ∞-cosmos $\mathcal{K}$)[7]

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[6] Emily Riehl (2014). *Categorical Homotopy Theory*. New Mathematical Monographs 24. Cambridge University Press. URL: https://math.jhu.edu/~eriehl/cathtpy/.

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[8] Claudio Hermida and Bart Jacobs (1997). "Structural Induction and Coinduction in a Fibrational Setting". In: *Information and Computation* 145.

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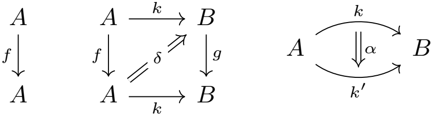
- Define 2-category $\mathcal{C}^{\circlearrowleft}$ of endomorphisms, distributive laws and distributive law morphisms with forgetful 2-functor $U : \mathcal{C}^{\circlearrowleft} \to \mathcal{C}$



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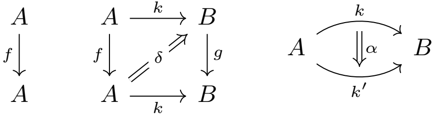

Theorem

If the 2-category $\mathcal{C}$ has a choice of coalgebra objects for all endomorphisms, then there is a product-preserving 2-functor $\mathrm{CoAlg}: \mathcal{C}^{\circlearrowleft} \to \mathcal{C}$ with a 2-natural transformation $p: \mathrm{CoAlg} \to U$.

[6] Emily Riehl (2014). *Categorical Homotopy Theory*. New Mathematical Monographs 24. Cambridge University Press. URL: https://math.jhu.edu/~eriehl/cathtpy/.

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Very useful consequences

Many known results are instances of 2-functoriality

- monoidal structure on coalgebras: present lax monoidal functor as distributive law
- transport of adjunctions
- determinisation: distributive laws of monads
- adequacy of coalgebraic modal logic: semantics via distributive law
- enriched coalgebraic modal logic via enriched fibrations

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Weakening product structure:

Theorem

If $\mathcal{K}$ is a (symmetric) monoidal 2-category, then $\mathcal{K}^{\circlearrowleft}$ is (symmetric) monoidal, U is strict monoidal, CoAlg is a lax monoidal functor and $p\colon \mathrm{CoAlg} \to U$ a monoidal natural transformation.

Colimits

For an appropriate 2-categorical definition of colimit we get a known result in general:

Definition

Let $\mathcal{C}$ be a Cartesian closed 2-category. A colimit of a morphism $d\colon D \to A^J$ parameterised in D and of shape J in $\mathcal{C}$ is a absolute left lifting (c, η) of d through diagonal $\Delta\colon A \to A^J$.

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Example (Instances)

- $\mathbf{Cat}$ — colimits in categories
- $\mathcal{V}$-$\mathbf{Cat}$ — conical colimits in categories enriched over $\mathcal{V}$ Cartesian closed
- $\mathbf{qCat_2}$ — homotopy colimits in quasi-categories
- $\mathfrak{h}\mathcal{K}$ — homotopy colimits in higher categories modelled by categories enriched over ∞-cosmos $\mathcal{K}$

Homotopical Coalgebra

Behaviour via Coalgebras

- Behaviour from repeated observation of a space X via map $c\colon X \to FX$
- Functor $F\colon \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$ determines the type of observations

Example (Hybrid Systems as Coalgebras)

- Hybrid system as coalgebra of trajectories in a space[9]
- $\mathbf{CG}$ category of compactly generated spaces[10] ("convenient")
- Define a functor $H\colon \mathbf{CG} \to \mathbf{CG}$ by

$$HX = \{(\varrho, d) \in X^{\mathbb{R}_{\geq 0}} \times [0, \infty] \mid \varrho \circ \min(-, d) = \varrho\}$$

- A coalgebra $c\colon X \to HX$ continuously assigns to $x \in X$ a pair (ϱ, d) of trajectory $\varrho\colon \mathbb{R}_{\geq 0} \to X$ that is constant after duration d.
- Can be refined to ensure that $c(x)$ starts at x

[9] Renato Neves et al. (Aug. 1, 2016). "Continuity as a Computational Effect". In: *Journal of Logical and Algebraic Methods in Programming*. Articles Dedicated to Prof. J. N. Oliveira on the Occasion of His 60th Birthday 85 (5, Part 2).

[10] J. Peter May (Sept. 1999). *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press. URL: https://www.math.uchicago.edu/~may/CONCISE/.

General Idea of Homotopical Coalgebra

Homomorphism up to homotopy

$$
\begin{array}{ccc}
X & \xrightarrow{c} & FX \\
{\scriptstyle f}\downarrow & \sim & \downarrow{\scriptstyle Ff} \\
Y & \xrightarrow[d]{} & FY
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
X & \xrightarrow{c} & FX & \xrightarrow{Fc} & F(FX) & \xrightarrow{F(Fc)} & \cdots \\
{\scriptstyle f}\downarrow & \sim & \downarrow{\scriptstyle Ff} & \sim & \downarrow{\scriptstyle F(Ff)} & \sim & \\
Y & \xrightarrow{d} & FY & \xrightarrow{Fd} & F(FY) & \xrightarrow{F(Fd)} & \cdots
\end{array}
$$

- Diagrams commute up to coherent homotopy
- Homomorphism f preserves and reflects behaviour up to coherent homotopy
- Inspired by coalgebra[11] and higher algebra[12]
- Realise homotopy coherence in $(\infty, 1)$-categories
- Homotopy (co)limits, obstruction theory via (co)homology, homotopy-invariant modal logic, ...

[11] Jan Rutten (2000). "Universal Coalgebra: A Theory of Systems". In: *Theor. Comput. Sci.* 249.1.

[12] Jacob Lurie (Sept. 2017). *Higher Algebra*. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

General Idea of Homotopical Coalgebra

Homomorphism up to homotopy

$$\begin{array}{ccc} X & \xrightarrow{c} & FX \\ f\downarrow & \sim & \downarrow Ff \\ Y & \xrightarrow[d]{} & FY \end{array} \qquad \begin{array}{ccccccc} X & \xrightarrow{c} & FX & \xrightarrow{Fc} & F(FX) & \xrightarrow{F(Fc)} & \cdots \\ f \downarrow & \sim & \downarrow Ff & \sim & \downarrow F(Ff) & \sim & \\ Y & \xrightarrow{d} & FY & \xrightarrow{Fd} & F(FY) & \xrightarrow{F(Fd)} & \cdots \end{array}$$

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Homotopy Coherence via Topological Enrichment

Topological Enrichment over Compactly Generated Spaces $\mathbf{CG}$

$\underline{\mathcal{C}}$ is a $\mathbf{CG}$-enriched category if

- it has a hom-space $\underline{\mathcal{C}}(X, Y) \in \mathbf{CG}$ for all objects X, Y
- composition are continuous maps $c_{X,Y,Z} \colon \underline{\mathcal{C}}(Y, Z) \times \underline{\mathcal{C}}(X, Y) \to \underline{\mathcal{C}}(X, Z)$
- there is an identity $\mathrm{id}_X \colon * \to \underline{\mathcal{C}}(X, X)$ for all objects X
- an associativity and two unit diagrams commute

Enrichment (plus other things) enables homotopy theory[13]

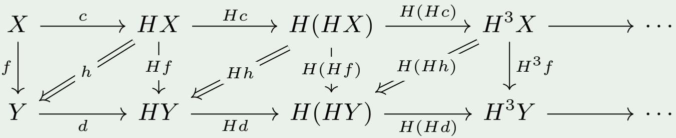
- Define a homotopy $h \colon f \Rightarrow g$ between $f, g \in \underline{\mathcal{C}}(X, Y)$ to be a continuous map $h \colon [0, 1] \to \underline{\mathcal{C}}(X, Y)$ with $h(0) = f$ and $h(1) = g$
- Write $f \sim g$ if there is some homotopy $f \Rightarrow g$
- Homotopy coherent nerve yields $(\infty, 1)$-category

[13] Emily Riehl (2014). *Categorical Homotopy Theory*. New Mathematical Monographs 24. Cambridge University Press. URL: https://math.jhu.edu/~eriehl/cathtpy/; Michael Shulman (June 30, 2009). *Homotopy Limits and Colimits and Enriched Homotopy Theory*. arXiv: math/0610194. preprint.

Behaviour up to Homotopy

Example

- Continuous maps form a space $\underline{\mathbf{CG}}(X, Y)$ and composition is continuous
- This makes $\underline{\mathbf{CG}}$ a $\mathbf{CG}$-enriched category
- Call $f\colon X \to Y$ a homotopical coalgebra morphism from $c\colon X \to HX$ to $d\colon Y \to HY$ if it comes with a homotopy $h\colon Hf \circ c \Rightarrow d \circ f$
- The functor H is $\mathbf{CG}$-enriched, that is, $H_{X,Y}\colon \mathbf{CG}(X, Y) \to \mathbf{CG}(HX, HY)$ is continuous
- Hence, homotopy $h : f \to g$ can be mapped to a homotopy $Hh\colon Hf \to Hg$ by $Hh = H_{X,Y} \circ h$
- Obtain a sequence of homotopies



Obstructions and Modal Logic

Example (Detecting Zeno Behaviour)

- Physically non-realisable behaviour
- For instance, infinitely fast or accelerating switching
- Detect as homotopical obstructions to coalgebra mapping problem
- Much easier to allow coalgebra homomorphism up to homotopy!

[14] Christina Vasilakopoulou (July 6, 2018). *On Enriched Fibrations*. arXiv: 1801.01386. preprint.

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Example (Homotopy-Invariant Logic)

- Fibration $p\colon \mathbf{cPred} \to \mathbf{CG}$ of closed predicates
- $\mathbf{CG}$-enriched in a suitable sense[14]
- Cartesian lifting of homotopy homomorphism yields homotopies between predicate transformers
- Axioms must account for these

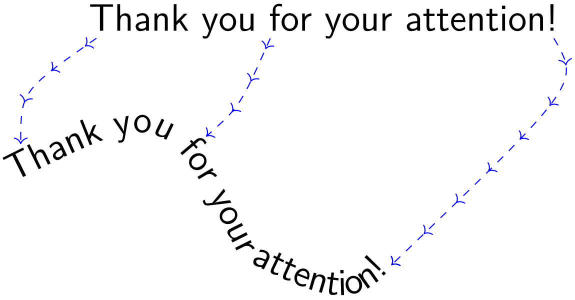
[14] Christina Vasilakopoulou (July 6, 2018). *On Enriched Fibrations*. arXiv: 1801.01386. preprint.

Wrapping Up

Outlook

1. Improve colimit results: non-conical colimits and non-Cartesian enrichment
2. Coalgebras on simplicial sets and topological spaces for concurrency and epistemic logic
3. Reconciliation with directed approaches[15]
4. Integrate with homotopical/model categories (enriched homotopy theory)
5. Enrichment is good for computation, but theory is simpler over quasi-categories
6. Homotopy-invariant coalgebraic modal logic
7. Obstruction theory via (co)homology

[15] Jérémy Dubut, Eric Goubault, and Jean Goubault-Larrecq (2016). "The Directed Homotopy Hypothesis". In: *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*. Ed. by Jean-Marc Talbot and Laurent Regnier. Vol. 62. LIPIcs. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.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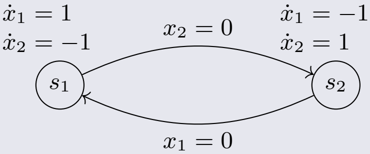
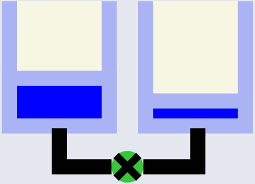
Zeno Behaviour Example

Zeno Behaviour

Sisyphus pumps water

- Two water tanks connected by a pump
- Pumps water until tank is empty and then switches direction
- Two states for the pumping directions
- Guards enable transitions
- Two sets of differential equations for linear flow





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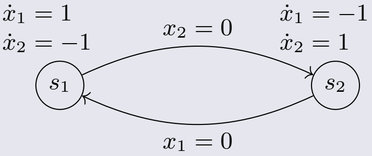
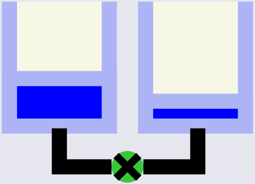




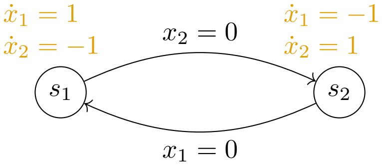
Not physically realisable

Infinite switching in finite time when both tanks are empty

Modelling the Water Tanks

Domains and guards

$$\Omega_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_i \geq 0\}$$
$$\Omega_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_i \geq 0\}$$
$$G_1 = \{(x_1, x_2) \in \Omega_1 \mid x_2 = 0\}$$
$$G_2 = \{(x_1, x_2) \in \Omega_2 \mid x_1 = 0\}$$



Modelling the Water Tanks

Domains and guards

$$\Omega_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_i \geq 0\}$$
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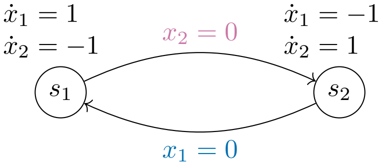

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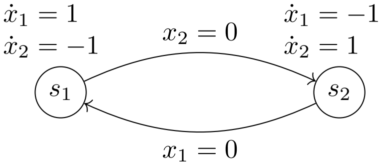
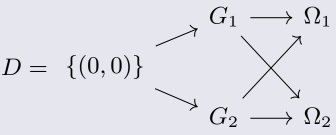
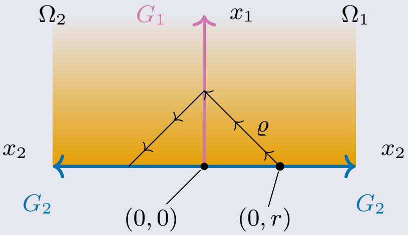


Hybrid computation as coalgebra on colimit space



$$S_1 = \operatorname{colim} D$$
$$c_1 : S_1 \to HS_1$$



Realisable Sisyphus

Switching

- Switching takes time
- But it is irrelevant how much
- Trajectories in homotopy colimit $\operatorname{hcolim} D$ of D!

$$S_2 = \operatorname{hcolim} D$$
$$c_2 \colon S_2 \to HS_2$$

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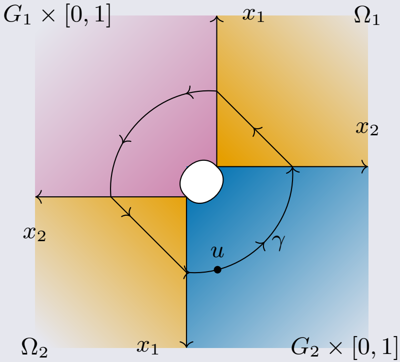

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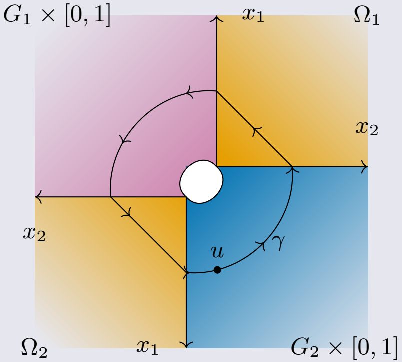

Postulate

Any physically realisable model must have a coalgebra map up to homotopy into c_2.

Homotopical Obstruction to Realisability

Water tank pump not realisable

- Let $f\colon S_1 \to S_2$ be a map with a homotopy $h\colon c_2 \circ f \Rightarrow Hf \circ c_1$ (endpoint-preserving)

[16] Tyler Westenbroek et al. (Jan. 1, 2021). "Smooth Approximations for Hybrid Optimal Control Problems with Application to Robotic Walking". In: *IFAC-PapersOnLine*. 7th IFAC Conference on Analysis and Design of Hybrid Systems ADHS 2021 54.5.

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- This allows us to show that any loop in S_2 can be contracted to a constant path
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Dual use

The other way around: c_2 forces system to be realisable[16]

[16] Tyler Westenbroek et al. (Jan. 1, 2021). "Smooth Approximations for Hybrid Optimal Control Problems with Application to Robotic Walking". In: *IFAC-PapersOnLine*. 7th IFAC Conference on Analysis and Design of Hybrid Systems ADHS 2021 54.5.

Homotopy-Invariant Modal Logic

Modal Logic on HDA

Show modalities and homotopy axiom[17]

$$\varphi ::= p \mid \bot \mid \varphi \to \varphi \mid \Diamond^{\uparrow}\varphi \mid \Diamond^{\downarrow}\varphi$$

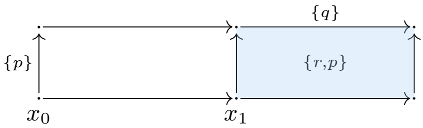
- $\Diamond^{\uparrow}\varphi$ holds if some action can be started and φ holds during execution
- $\Diamond^{\downarrow}\varphi$ holds if some action can be ended and φ holds afterwards

Interpretation over an HDA with cubes X

$$\begin{aligned}
[\![\Diamond^{\uparrow}\varphi]\!]_n &= \{x \in X_n \mid \exists x' \in X_{n+1}.\, x \text{ is a boundary of } x' \text{ and } x' \in [\![\varphi]\!]_{n+1}\} \\
[\![\Diamond^{\downarrow}\varphi]\!]_{n+1} &= \{x \in X_{n+1} \mid \exists x' \in X_n.\, x' \text{ is a boundary of } x \text{ and } x' \in [\![\varphi]\!]_n\} \\
x \vDash \varphi &\iff \exists n.\, x \in [\![\varphi]\!]_n
\end{aligned}$$

[17] Cristian Prisacariu (2010). "Modal Logic over Higher Dimensional Automata". In: *Proc. of CONCUR 2010*.

Homotopy-Invariance for HDA Logic



Example

$$x_0 \vDash \Diamond^{\uparrow} p \qquad x_1 \vDash \Diamond^{\uparrow}\Diamond^{\uparrow}\Diamond^{\downarrow} q$$
$$x_1 \vDash \Diamond^{\uparrow}\Diamond^{\uparrow} r \wedge p \qquad x_1 \vDash \Diamond^{\uparrow}\Diamond^{\downarrow}\Diamond^{\uparrow} q$$

Interchange Axioms[18]

$$\Diamond^{\uparrow}\Diamond^{\uparrow}\Diamond^{\downarrow}\varphi \to \Diamond^{\uparrow}\Diamond^{\downarrow}\Diamond^{\uparrow}\varphi \tag{A10}$$
$$\Diamond^{\uparrow}\Diamond^{\downarrow}\Diamond^{\downarrow}\varphi \to \Diamond^{\downarrow}\Diamond^{\uparrow}\Diamond^{\downarrow}\varphi \tag{A10'}$$

[18] Cristian Prisacariu (2014). *Higher Dimensional Modal Logic*. arXiv: 1405.4100. URL: http://arxiv.org/abs/1405.4100. preprint.

Coalgebraic Modal Logic

One view based on dual adjunctions, so-called logical connections[19]

$$F \circlearrowright \mathcal{C} \underset{Q}{\overset{P}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}^{\mathrm{op}} \circlearrowleft L^{\mathrm{op}} \quad \text{and} \quad \varrho\colon PF \to L^{\mathrm{op}}P \quad \text{and} \quad \alpha\colon L\Phi \to \Phi$$

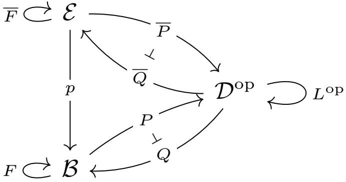
Components

- $\mathcal{C}$ category for "states" in coalgebras
- F behaviour functor to get coalgebras $X \to FX$
- $\mathcal{D}$ typically category of algebras for logical operators
- L specifies modal operators
- initial algebra α for syntax
- distributive law $\varrho\colon LP \to PF$ to give semantics of formulas in a coalgebra
- $P \dashv Q$ is often concrete duality by mapping into dualising object

[19] Dusko Pavlovic, Michael W. Mislove, and James Worrell (2006). "Testing Semantics: Connecting Processes and Process Logics". In: *Proceedings of Algebraic Methodology and Software Technology, 11th International Conference, AMAST 2006*. Ed. by Michael Johnson and Varmo Vene. Vol. 4019. Lecture Notes in Computer Science. Springer; Toby Wilkinson (2013). "Enriched Coalgebraic Modal Logic". PhD thesis. URL: http://eprints.soton.ac.uk/354112/.

Modal Logic for General Coinductive Predicates

Previous picture is restricted to logic for behavioural equivalence/bisimilarity!



Components[20]

- $p\colon \mathcal{E} \to \mathcal{B}$ fibration
- coalgebras for $\overline{F}$ are proofs of coinductive predicates
- final coalgebras in fibres are typically called coinductive predicates
- soundness (adequacy) and completeness (expressiveness) results provable in this setting

[20] Clemens Kupke and Jurriaan Rot (Dec. 15, 2021). "Expressive Logics for Coinductive Predicates". In: *Logical Methods in Computer Science* Volume 17, Issue 4.

Higher Coalgebraic Modal Logic

- Theorem from earlier gives adequacy in categories
- Reason is that 2-categorically defined Cartesian fibrations are the right thing
- In quasi-categories this fails
- Needs some work directly with quasi-categories[21]
- Develop coalgebraic modal logic further in higher categories

[21] Emily Riehl and Dominic Verity (2022). *Elements of ∞-Category Theory*. Cambridge University Press (CUP).