CUP and Garlic: Coinductive Uniform Proofs and Guarded Recursive Logic

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A Workshop for John Power, Bath
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Continuing the Timeline

- 2015: ALCOP Visit in Dundee
- 2016: CoALP-Ty
- 2017: Workshop at Defence
- 2018: Visit at HW
- 2019: ESOP Proof Relevance
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CUP and Garlic
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CUP and Garlic

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Proof Relevance
Foundation for Proof Search in Coinductive Theories

- Resolution is a great tool for search
- But we need to understand its power and limitations
- Approach: Devise two proof systems
  1. Coinductive Uniform Proofs (cut-free, operational proof search)
  2. Guarded Recursive First-Order Logic (principled semantics)
Garlic: Guarded Recursive Logic
Recursive Proofs

- Recursion as first step to proof search
- Eliminates the need to find invariants:

\[ \varphi \Rightarrow \psi[\varphi/X] \]

\[ \varphi \Rightarrow \nu X. \psi \]

- Recursion will be controlled by the so-called later modality
Formulas

Formulas of Garlic

\[ \phi, \psi ::= p \ M_1 \cdots M_n, \quad p \text{ predicate symbol}, \ M_k \text{ term} \]

\[ \top \ | \ \phi \land \psi \ | \ \phi \lor \psi \ | \ \phi \rightarrow \psi \ | \ \forall x : \tau. \ \phi \ | \ \exists x : \tau. \ \phi \]

\[ \top \]

\[ \top \]

\[ \top \]
## Proof System

**Standard First-Order Intuitionistic Logic plus**

### Rules for the later modality

\[
\begin{align*}
\frac{\Gamma, \Delta \vdash \varphi}{\Gamma, \Delta \vdash \Box \varphi} & \quad \text{(Next)} \\
\frac{\Gamma, \Delta \vdash (\varphi \rightarrow \psi)}{\Gamma, \Delta \vdash \Box (\varphi \rightarrow \psi)} & \quad \text{(Mon)} \\
\frac{\Gamma, \Delta, \Box \varphi \vdash \varphi}{\Gamma, \Delta \vdash \varphi} & \quad \text{(Löb)}
\end{align*}
\]

### Axioms for coinductive Horn clause theories \( P \)

\[
\begin{align*}
\forall \overrightarrow{x} \cdot (A_1 \land \cdots \land A_n) & \rightarrow B \in P \\
\frac{\Gamma, \Delta \vdash \forall \overrightarrow{x} \cdot (\Box A_1 \land \cdots \land \Box A_n) \rightarrow B}{\Gamma, \Delta \vdash \forall \overrightarrow{x} \cdot (\Box A_1 \land \cdots \land \Box A_n) \rightarrow B}
\end{align*}
\]
Semantics of Garlic¹

¹Yummy!
Fibration of Descending Chains

- $p: \mathbf{E} \to \mathbf{B}$ fibration
- $\mathbf{I}$ well-founded index category
- Write $\mathbf{C}$ for category $[\mathbf{I}^{\text{op}}, \mathbf{C}]$ of functors (diagrams) $\mathbf{I}^{\text{op}} \to \mathbf{C}$
- Gives fibred functor

\[
\begin{array}{ccc}
\text{Fib} & \xleftarrow{(-)} & \text{Fib} \\
\downarrow & & \downarrow \\
\text{Cat} & & \text{Cat}
\end{array}
\]

- $\leftarrow p: \leftarrow \mathbf{E} \to \leftarrow \mathbf{B}$ fibration of descending $\mathbf{I}$-indexed chains
- Preserves change-of-base (pullback)
Descending Chains

- An $\omega^{\text{op}}$-chain $\sigma$ above $c$ in a fibration $p$ is given by

$$
\begin{align*}
\text{E} : & \quad \sigma_0 \leftarrow \sigma_1 \leftarrow \sigma_2 \leftarrow \cdots \\
& \quad \downarrow p \quad \downarrow p \quad \downarrow p \\
\text{B} : & \quad c_0 \leftarrow c_1 \leftarrow c_2 \leftarrow \cdots
\end{align*}
$$

- Applying the later modality gives

$$
\begin{align*}
\blacktriangleright \sigma : & \quad 1 \leftarrow \sigma_0 \leftarrow \sigma_1 \leftarrow \sigma_2 \leftarrow \cdots \\
& \quad \downarrow p \quad \downarrow p \quad \downarrow p \\
\blacktriangleright c : & \quad 1 \leftarrow c_0 \leftarrow c_1 \leftarrow c_2 \leftarrow \cdots
\end{align*}
$$

- General well-founded index (akin to inflationary fixed points):

$$
(\blacktriangleright \sigma)_\alpha = \lim_{\beta < \alpha} \sigma_\beta \quad \overset{p}{\mapsto} \quad (\blacktriangleright c)_\alpha = \lim_{\beta < \alpha} c_\beta
$$
Constant Index Chains

- Fixing the index of chains: change-of-base along constant functor chain $K^B$ given by $K^B(I)_\alpha = I$

$$
\begin{align*}
\overline{E} & \longrightarrow \overline{E} \\
\downarrow \overline{p} & \quad \quad \downarrow \overline{p} \\
B & \longrightarrow B
\end{align*}
$$

- $\overline{p}$ is keeps logical structure of $p$ (under some conditions)
- (Co)Products along morphism $u: X \rightarrow Y$ in $B$ lift to $\overline{p}$:

$$
\begin{align*}
\overline{E}_Y & \quad \overline{E}_X \\
\overline{E}_Y & \quad \overline{E}_X
\end{align*}
$$
Fibred Cartesian Closure

Theorem

The fibration $\overline{p}: \overline{E} \to B$ is a fibred CCC if $p$ has bounded $I^{\text{op}}$-limits and is a fibred CCC given by the end

$$(\sigma^\tau)_\alpha = \int_{\beta < \alpha} (\sigma_\beta)^{\tau_\beta}.$$ 

- Resembles to Kripke semantics for intuitionistic implication
- Captures the space of causal maps $\lim \tau \to \lim \sigma$
- Note: Theorem holds more generally for chains with non-constant index and for index categories that are not well-founded orders, see paper at CALCO’19.
Fibred Cartesian Closure

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Constant Index Chains

- $\bar{p}$ fibred Cartesian closed category
- fibred functor with next: $\text{Id} \Rightarrow \triangleright$ and $\text{löb}_\sigma : \sigma \triangleright \sigma \rightarrow \sigma$
- $K^E$ is faithful
- If $p$ is first-order fibration, then $\bar{p}$ is FO fibration and $K^E$ is full
- FO fibration results restrict to any subset of connectives
  - $L^E$ exists if fibred $\Gamma^{\text{op}}$-limits exist in $p$
  - $L^E$ preserves FO structure, if $\vee$ and $\bigsqcup_u$ preserve $\Gamma^{\text{op}}$-limits
**Constant Index Chains**

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Final Coalgebras via Locally Contractive Functors

- Functor $\Phi : E_I \rightarrow E_I$ from logic program (backwards closure)
- $(\triangleright \circ \Phi) : \overline{E}_I \rightarrow \overline{E}_I$, where $\Phi$ is point-wise application:
  $$(\triangleright \Phi)(\sigma) = 1 \leftarrow \Phi(\sigma_0) \leftarrow \Phi(\sigma_1) \leftarrow \Phi(\sigma_2) \leftarrow \cdots$$
- Final chain is unique fixed point of $\triangleright \Phi$ in $\overline{E}_I$
- Soundness of unfolding along clauses in theory

\[
\forall \vec{x}. (A_1 \land \cdots \land A_n) \rightarrow B \in P
\]

\[
\Gamma \mid \Delta \vdash \forall \vec{x}. (\triangleright A_1 \land \cdots \land \triangleright A_n) \rightarrow B
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• Functor $\Phi : E_I \rightarrow E_I$ from logic program (backwards closure)
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\Gamma \mid \Delta \vdash \forall \vec{x}. (\triangleright A_1 \land \cdots \land \triangleright A_n) \rightarrow B
\]
Replacing Limits by Chains

- The limit adjunction lifts, if $\Phi: E_I \to E_I$ preserves $I^{\text{op}}$-limits

\[ \text{CoAlg}(\Phi) \xleftarrow{\perp} \text{CoAlg}(\Phi^{\text{op}}) \]

- Reasoning on chains instead of Herbrand model
Replacing Limits by Chains

- The limit adjunction lifts, if $\Phi : E \to E$ preserves $I^{\text{op}}$-limits

- Reasoning on chains instead of Herbrand model
Coinductive Uniform Proofs
What and Why Uniform Proofs?

Issues with Garlic

- Recursion can be started anywhere
- Proof system has cut rule (through implication)
- \( \Rightarrow \) Prevents algorithmic proof search

Towards proof search

- Fix where recursion can start
- Eliminate cut, while preserving implication
- Operational semantics for proofs that correspond to resolution
- \( \Rightarrow \) Proof search is semi-decidable resolution strategy
What and Why Uniform Proofs?

### Issues with Garlic

- Recursion can be started anywhere
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- $\implies$ Prevents algorithmic proof search

### Towards proof search

- Fix where recursion can start
- Eliminate cut, while preserving implication
- Operational semantics for proofs that correspond to resolution
- $\implies$ Proof search is semi-decidable resolution strategy
Starting a coinductive uniform proof

\[ \Sigma; P; \varphi \Rightarrow \langle \varphi \rangle \]

\[ \Sigma; P \not\Rightarrow \varphi \quad \text{CO-FIX} \]

Controlling the use of the coinduction hypothesis

\[ \Sigma; P \cup \Delta \xrightarrow{D} A \quad D \in P \]

\[ \Sigma; P; \Delta \Rightarrow \langle A \rangle \quad \text{DECIDE} \langle \rangle \]

\[ \Sigma; P; \Delta, \varphi_1 \Rightarrow \langle \varphi_2 \rangle \quad \text{\rightarrow R} \langle \rangle \]

\[ \Sigma; P; \Delta \Rightarrow \langle \varphi_1 \rightarrow \varphi_2 \rangle \]

+ interaction of \( \langle \neg \rangle \) with \( \land, \forall \) + standard uniform proof rules
Coinductive Uniform Proofs (CUP)

Starting a coinductive uniform proof

\[ \Sigma; P; \varphi \Rightarrow \langle \varphi \rangle \]

\[ \Sigma; P \not\Rightarrow \varphi \text{ CO-FIX} \]

Controlling the use of the coinduction hypothesis

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\[ \Sigma; P; \Delta, \varphi_1 \Rightarrow \langle \varphi_2 \rangle \]

\[ \Sigma; P; \Delta \Rightarrow \langle \varphi_1 \rightarrow \varphi_2 \rangle \rightarrow R \]

+ interaction of \( \langle - \rangle \) with \( \wedge, \forall \)

+ standard uniform proof rules
Semantics and Proof Translation

\[ \mathcal{L}^P_{\text{CUP}} \xrightarrow{T} \mathcal{L}^P_{\text{Garlic}} \xrightarrow{\pi} \text{Pred} \]

\[ \mathcal{L}^P_{\text{Garlic}} \xrightarrow{\pi_{\text{Garlic}}} \text{Set} \]

\[ \mathcal{C} \xrightarrow{\pi_{\text{CUP}}} \text{Pred} \]

\[ \frac{\llbracket - \rrbracket^P}{L} \]

\[ \frac{\perp}{K} \]

\[ \frac{\text{Set}}{\pi} \]

\[ \frac{\text{Pred}}{\overline{\pi}} \]
Semantics and Proof Translation

- $\cal{C}$ — Contexts and guarded terms
- $\mathcal{L}_C^P$ — Formulas and provability in CUP relative to $P$
- $\mathcal{L}_G^P$ — Formulas and provability in Garlic relative to $P$
- Pred — Predicate fibration
- $\overline{\text{Pred}}$ — Descending chains of predicates (Kripke model)
Semantics and Proof Translation

- $\mathcal{L}^P_{CUP}$ — Formulas and provability in CUP relative to $P$
- $\mathcal{L}^P_{Garlic}$ — Formulas and provability in Garlic relative to $P$
- Pred — Predicate fibration
- Pred — Descending chains of predicates (Kripke model)

- $\mathcal{C}$ — Contexts and guarded terms
Semantics and Proof Translation

\( \mathcal{L}_{\text{CUP}}^P \rightarrow \mathcal{L}_{\text{Garlic}}^P \rightarrow \mathcal{L}_{\text{Pred}} \rightarrow \mathcal{L}_{\text{Pred}} \rightarrow \mathcal{L}_{\text{Pred}} \rightarrow \mathcal{L}_{\text{Pred}} \)

- \( \mathcal{C} \) — Contexts and guarded terms
- \( \mathcal{L}_{\text{CUP}}^P \) — Formulas and provability in CUP relative to \( P \)
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Henning
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Semantics and Proof Translation

- $\llbracket \cdot \rrbracket$ — Semantics of types and guarded terms
- $\llbracket \cdot \rrbracket^P$ — Semantics of formulas and soundness
- $T$ — Proof translation
- Lifting $K \vdash L$ — Soundness and partial completeness
- NB: $\mathcal{L}_\text{CUP}^P$ is not a category, as CUP lacks cut
Semantics and Proof Translation

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Semantics and Proof Translation

\[ \mathcal{L}_\text{CUP}^P \xrightarrow{T} \mathcal{L}_\text{Garlic}^P \xrightarrow{\pi} \text{Set} \]

- \([-]\) — Semantics of types and guarded terms
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Extending the Timeline

- Automation and lemma generation
- Completeness and admissible cut rule
- Generate proof objects (propositions-as-types interpretation)
- Inductive-coinductive Horn clause theories
- Typed Horn clauses and type extraction for logic programs
- How can we go beyond causality?
Thank you!