Recursive Proofs for Coinductive Predicates

The Later Modality in Fibrations

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Motivation

Original Motivation

- Syntactic logic for program equivalence (in my thesis)
- Recursive proof system based on later modality
- Recursion gives rise to proof search
- Construction of the logic, and proofs of meta properties and soundness are pedestrian
- Need for an abstract framework

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General Approach

Provide framework to

- Extend logic with guarded recursion
- Reason about arbitrary coinductive predicates
- Use any (compatible) up-to technique in proofs

Related Systems

- *Topos of trees* (Birkedal, Møgelberg, et al.) — We generalise this to fibrations; requires encoding of a logic and does not directly yield a new syntax
- *Parameterised coinduction* (Moss and Hur et al.) — not developed for first- and higher-order logics
- *CIRC* (Roşu, Lucanu) — cyclic proof system for bisimilarity; hard to understand and hand-crafted
- *Cyclic proof systems* — hand-crafted; global correctness conditions
- *(Bisimulation) Games* — global parity conditions
- *Step-indexed relations* (Appel, McAllester and Birkedal, Møgelberg et al.) — instance of the framework we develop ☺

Stream Differential Equations

Head and Tail

$$s : \mathbb{R}^\omega \qquad s_0 : \mathbb{R} \qquad s' : \mathbb{R}^\omega$$

Example (Constant Streams)

$$a^\omega : \mathbb{R}^\omega \qquad a^\omega_0 = a \qquad (a^\omega)' = a^\omega$$

Example (Point-wise Stream Addition)

$$\oplus : \mathbb{R}^\omega \to \mathbb{R}^\omega \to \mathbb{R}^\omega$$

$$(s \oplus t)_0 = s_0 + t_0 \qquad (s \oplus t)' = s' \oplus t'$$

Example (Stream of Positive Numbers)

$$s : \mathbb{R}^\omega \qquad s_0 = 1 \qquad s' = 1^\omega \oplus s$$

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Point-wise Positive Streams

Example (Predicate Transformer)

$$\Phi(P \subseteq \mathbb{R}^\omega) = \{s \in \mathbb{R}^\omega \mid s_0 > 0 \land s' \in P\}$$

- Φ monotone
- Greatest fixed point $\nu\Phi$ exists
- $s \in \nu\Phi$ iff s is point-wise greater than 0

Positive Numbers are Greater Than 0

$$\dfrac{\dfrac{\textbf{(Next)}\;\dfrac{\text{(Def. of }s\text{)}\;\dfrac{\dfrac{}{\vdash 1 > 0}}{\vdash s_0 > 0}}{\vdash \blacktriangleright(s_0 > 0)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\blacktriangleright\varphi \vdash \blacktriangleright\varphi}}{\blacktriangleright\varphi \vdash \blacktriangleright(s \in \nu\Phi)}\;\textbf{(Pr)}}{\blacktriangleright\varphi \vdash \blacktriangleright(1^\omega \oplus s \in C(\nu\Phi))}\;\text{(Def. }C\text{)}}{\blacktriangleright\varphi \vdash \blacktriangleright(1^\omega \oplus s \in \nu\Phi)}\;\text{(}C\text{ compat.)}}{\blacktriangleright\varphi \vdash \blacktriangleright(s' \in \nu\Phi)}\;\text{(Def. of }s\text{)}}{\dfrac{\blacktriangleright\varphi \vdash \blacktriangleright(s_0 > 0 \wedge s' \in \nu\Phi)}{\dfrac{\blacktriangleright\varphi \vdash s \in \nu\Phi}{\vdash s \in \nu\Phi}\;\textbf{(Löb)}}\;\textbf{(Step)}}\;(\blacktriangleright\text{ preserves }\wedge)$$

Inference Rule

Positive Numbers are Greater Than 0

$$
\dfrac{
  \text{(Next)}\;\dfrac{\text{(Def. of } s\text{)}\;\dfrac{\overline{\vdash 1 > 0}}{\vdash s_0 > 0}}{\vdash \blacktriangleright(s_0 > 0)}
  \qquad
  \dfrac{\dfrac{\dfrac{\dfrac{\overline{\blacktriangleright\varphi \vdash \blacktriangleright\varphi}}{\blacktriangleright\varphi \vdash \blacktriangleright(s \in \nu\Phi)}\;\textbf{(Pr)}}{\blacktriangleright\varphi \vdash \blacktriangleright(1^\omega \oplus s \in C(\nu\Phi))}\;\text{(Def. } C\text{)}}{\blacktriangleright\varphi \vdash \blacktriangleright(1^\omega \oplus s \in \nu\Phi)}\;(C\text{ compat.})}{\blacktriangleright\varphi \vdash \blacktriangleright(s' \in \nu\Phi)}\;\text{(Def. of } s\text{)}
}{
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}\;(\blacktriangleright\text{ preserves } \land)
$$

Inference Rule

$$\varphi := s \in \nu\Phi$$

$$\dfrac{\Delta, \blacktriangleright\varphi \vdash \varphi}{\Delta \vdash \varphi}\;\textbf{(Löb)}$$

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Inference Rule

$$\begin{array}{c} \Delta \vdash \blacktriangleright(s \in \Phi(\nu\Phi)) \\ \hline\hline \Delta \vdash s \in \nu\Phi \end{array}\;\textbf{(Step)}$$

$$\Phi(P) = \{s \in \mathbb{R}^\omega \mid s_0 > 0 \wedge s' \in P\}$$

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$$
\dfrac{\dfrac{\dfrac{\blacktriangleright\varphi \vdash \blacktriangleright(s_0 > 0 \wedge s' \in \nu\Phi)}{}\!}{}}{}
$$

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\dfrac{\dfrac{\dfrac{\text{(Def. of } s\text{)}\ \dfrac{\dfrac{}{\vdash 1 > 0}}{\vdash s_0 > 0}}{\color{red}\vdash \blacktriangleright(s_0 > 0)}\ \textbf{(Next)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\blacktriangleright\varphi \vdash \blacktriangleright\varphi}}{\blacktriangleright\varphi \vdash \blacktriangleright(s \in \nu\Phi)}\ \textbf{(Pr)}}{\blacktriangleright\varphi \vdash \blacktriangleright(1^\omega \oplus s \in C(\nu\Phi))}\ (\text{Def. } C)}{\blacktriangleright\varphi \vdash \blacktriangleright(1^\omega \oplus s \in \nu\Phi)}\ (C \text{ compat.})}{\blacktriangleright\varphi \vdash {\color{red}\blacktriangleright(s' \in \nu\Phi)}}\ (\text{Def. of } s)}{\blacktriangleright\varphi \vdash \blacktriangleright(s_0 > 0 \wedge s' \in \nu\Phi)}\ {\color{red}(\blacktriangleright \text{ preserves } \wedge)}}{\dfrac{\blacktriangleright\varphi \vdash s \in \nu\Phi}{\vdash s \in \nu\Phi}\ \textbf{(Löb)}}\ \textbf{(Step)}
$$

Inference Rule

$$
\dfrac{\Delta \vdash \blacktriangleright\varphi \wedge \blacktriangleright\psi}{\Delta \vdash \blacktriangleright(\varphi \wedge \psi)}\ (\blacktriangleright \text{ preserves } \wedge)
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Inference Rule

$$\dfrac{\Delta \vdash \varphi}{\Delta \vdash \blacktriangleright\varphi}\ \textbf{(Next)}$$

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$$
\dfrac{C \text{ compatible up-to} \qquad \Delta \vdash t \in C(\nu\Phi)}{\Delta \vdash t \in \nu\Phi}\;\text{(C compatible)}
$$

$$
C(P \subseteq \mathbb{R}^\omega) = \{1^\omega \oplus s \mid s \in P\}
$$

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Inference Rule

$$\dfrac{\varphi \in \Delta}{\Delta \vdash \varphi}\ \textbf{(Pr)}$$

Idea

Extending a Logic

- Given a logic $\mathcal{L}$ with formulas φ and provability $\Gamma \mid \Delta \vdash \varphi$
- Construct a new logic $\overline{\mathcal{L}}$ with the same propositional and first-order connectives, . . .
- . . . and a new connective $\blacktriangleright$, the later modality, that fulfils the axioms for the later modality . . .
- . . . and enables coinductive predicates and up-to techniques

Rules for the later modality

$$\frac{\Gamma \mid \Delta \vdash \varphi}{\Gamma \mid \Delta \vdash \blacktriangleright \varphi}\ \textbf{(Next)} \qquad \frac{\Gamma \mid \Delta \vdash \blacktriangleright(\varphi \to \psi)}{\Gamma \mid \Delta \vdash \blacktriangleright \varphi \to \blacktriangleright \psi}\ \textbf{(Mon)}$$

$$\frac{\Gamma \mid \Delta, \blacktriangleright \varphi \vdash \varphi}{\Gamma \mid \Delta \vdash \varphi}\ \textbf{(Löb)}$$

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Rules for coinductive predicates and up-to techniques

$$\frac{\Gamma \mid \Delta \vdash \blacktriangleright(s \in \Phi(\nu\Phi))}{\Gamma \mid \Delta \vdash s \in \nu\Phi} \textbf{ (Step)}$$

(the rule is printed with a double line)

$$\frac{C \text{ is } \Phi\text{-compatible} \qquad \Gamma \mid \Delta \vdash t \in C(\nu\Phi)}{\Gamma \mid \Delta \vdash t \in \nu\Phi} \textbf{ (Up-to)}$$

Setup

Fibrations

- Fibrations provide abstraction of first- (and higher-)order logic
- $\mathbf{B}$ — Category of typed contexts and terms
- $\mathbf{E}$ — Category of formulas with variables typed in $\mathbf{B}$
- $p \colon \mathbf{E} \to \mathbf{B}$ — Functor that assigns to a formula its context
- $\mathbf{E}_I = p^{-1}(\mathrm{id}_I)$ — Fibre above I with formulas in context I
- $u^* \colon \mathbf{E}_J \to \mathbf{E}_I$ — Substitution functor for $u \colon I \to J$ in $\mathbf{B}$

Example

- Set-based predicates (and relations): $\mathrm{Pred} \to \mathbf{Set}$
- Quantitative predicates: $\mathrm{qPred} \to \mathbf{Set}$
- Syntactic logic over syntactic terms: $\mathcal{L} \to \mathcal{C}$
- Set-indexed families (dependent types): $\mathrm{Fam}(\mathbf{C}) \to \mathbf{Set}$

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Coinductive Predicates

Predicate lifting G of behaviour functor F

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "G"] \arrow[d, "p"] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{B} \arrow[r, "F"] & \mathbf{B}
\end{tikzcd}$$

commutes and G preserves Cartesian morphisms.

Predicate transformer for coalgebra $c\colon X \to FX$

$$\Phi := \mathbf{E}_X \xrightarrow{G} \mathbf{E}_{FX} \xrightarrow{c^*} \mathbf{E}_X$$

Coinductive predicate

Final Φ-coalgebra $\xi\colon \nu\Phi \to \Phi(\nu\Phi)$

Greater-Than-0 Example

Example (Predicate lifting and coinductive predicate)

$$F\colon \mathbf{Set} \to \mathbf{Set} \qquad G\colon \mathrm{Pred} \to \mathrm{Pred}$$

$$F = \mathbb{R} \times \mathrm{Id} \qquad G(X, P) = (FX, \{(a, x) \mid a > 0 \wedge x \in P\})$$

Predicate transformer: $\Phi = \langle \mathrm{hd}, \mathrm{tl} \rangle^* \circ G$

$$\Phi(P \subseteq \mathbb{R}^\omega) = \{s \in \mathbb{R}^\omega \mid s_0 > 0 \wedge s' \in P\}$$

Coinductive predicate: $\nu\Phi \subseteq \Phi(\nu\Phi)$

Example (Notation)

Given a descending chain $\sigma \in \overline{\mathrm{Pred}}_X$, we define

$$\vdash \sigma := \overline{\mathbf{1}}_X \sqsubseteq \sigma \qquad (\iff \text{there exists } \overline{\mathbf{1}}_X \to \sigma)$$

$$x \mathrel{\overline{\in}} \sigma := \sigma^{K_{\{x\}}}$$

$$\vdash s \mathrel{\overline{\in}} \overleftarrow{\Phi} \iff \forall n \in \mathbb{N}.\, s \in \overleftarrow{\Phi}_n \overset{Thm}{\iff} s \in \nu\Phi \iff s \text{ greater t. } 0$$

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ω-Presheaves:
A Habitat for the Later Modality

Category of Descending Chains

$\overline{\mathbf{C}} = [\omega^{\mathrm{op}}, \mathbf{C}] =$ “category of functors $\omega^{\mathrm{op}} \to \mathbf{C}$

and natural tranformations”

$$\begin{tikzcd}
\sigma_0 \arrow[dd, "f_0"'] & \sigma_1 \arrow[l, "\sigma(0\leq1)"'] \arrow[dd, "f_1"'] & \sigma_2 \arrow[l, "\sigma(1\leq2)"'] \arrow[dd, "f_2"'] & \sigma_3 \arrow[l, "\sigma(2\leq3)"'] \arrow[dd, "f_3"'] & {} \arrow[l, "\cdots"'] \\
 & & & & \cdots \\
\tau_0 & \tau_1 \arrow[l, "\tau(0\leq1)"'] & \tau_2 \arrow[l, "\tau(1\leq2)"'] & \tau_3 \arrow[l, "\tau(2\leq3)"'] & {} \arrow[l, "\cdots"']
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\end{tikzcd}$$

ω-Presheaves in Fibrations

Lemma

The functor $\overline{(-)}$ *is a fibred functor on the fibration* $\mathbf{Fib} \to \mathbf{Cat}$.

Fibred Sequences

$$\begin{tikzcd}
\overline{\mathbf{E}} \arrow[d, "\overline{p}"] & \sigma \arrow[d, mapsto] & \sigma_0 & \sigma_1 \arrow[l, "\sigma(0\leq1)"'] & \sigma_2 \arrow[l, "\sigma(1\leq2)"'] & \sigma_3 \arrow[l, "\sigma(2\leq3)"'] & {} \arrow[l, "\cdots"'] \\
\overline{\mathbf{B}} & c = p \circ \sigma & c_0 & c_1 \arrow[l, "c(0\leq1)"'] & c_2 \arrow[l, "c(1\leq2)"'] & c_3 \arrow[l, "c(2\leq3)"'] & {} \arrow[l, "\cdots"']
\end{tikzcd}$$

Coinductive Predicates as Chains

The Final Chain

- $K_X \in \overline{\mathbf{B}}$ constantly X chain
- Given $\Phi\colon \mathbf{E}_X \to \mathbf{E}_X$, define $\overleftarrow{\Phi} \in \overline{\mathbf{E}}_{K_X}$

$$\overleftarrow{\Phi} := \ \mathbf{1} \xleftarrow{\quad ! \quad} \Phi(\mathbf{1}) \xleftarrow{\Phi(!)} \Phi^2(\mathbf{1}) \xleftarrow{\Phi^2(!)} \Phi^3(\mathbf{1}) \xleftarrow{\quad \cdots \quad}$$

Theorem

If Φ preserves ω^{op}-limits, then maps $A \to \nu\Phi$ in $\mathbf{E}_X$ are the same as maps $K_A \to \overleftarrow{\Phi}$ in $\overline{\mathbf{E}}_{K_X}$:

$$\begin{array}{c} A \longrightarrow \nu\Phi \\ \hline\hline K_A \longrightarrow \overleftarrow{\Phi} \end{array}$$

Birds-Eye View on the Framework

- Given $p\colon \mathbf{E} \to \mathbf{B}$, produce a fibration $\overline{p}\colon \overline{\mathbf{E}} \to \overline{\mathbf{B}}$, s.t.
- connectives (fibred products and coproducts, and quantifiers) in p induce connectives in $\overline{p}$
- fibred exponents and finite limits in p induce exponents in $\overline{p}$
- the later modality is a fibred functor on $\overline{p}$ and preserves products, exponents and universal quantifiers
- we can prove coinductive predicates and use up-to techniques

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Rules for the later modality

$$\frac{f\colon \tau \to \sigma}{\blacktriangleright f\colon \blacktriangleright \tau \to \blacktriangleright \sigma}\ \textbf{(Mon)} \qquad \frac{f\colon \tau \to \sigma}{\mathrm{next}_\sigma \circ f\colon \tau \to \blacktriangleright \sigma}\ \textbf{(Next)}$$

$$\frac{f\colon \tau \times \blacktriangleright \sigma \to \sigma}{\mathrm{l\ddot{o}b}_\sigma \circ \lambda f\colon \tau \to \sigma}\ \textbf{(Löb)}$$

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Rules for coinductive predicates and up-to techniques

$$\begin{array}{c} f : \tau \to \blacktriangleright\left(\overline{\Phi}\overleftarrow{\Phi}\right) \\ \hline\hline f : \tau \to \overleftarrow{\Phi} \end{array}\ \textbf{(Step)}$$

$$\begin{array}{c} f : \tau \to \overline{T}\overleftarrow{\Phi} \qquad \rho\colon T\Phi \Rightarrow \Phi T \ \ (T \text{ compatible}) \\ \hline \overleftarrow{\rho} \circ f : \tau \to \overleftarrow{\Phi} \end{array}\ \textbf{(Up-to)}$$

Later Modality

Theorem

For each $c \in \overline{\mathbf{B}}$, there is a fibred functor $\blacktriangleright^c \colon \overline{\mathbf{E}}_c \to \overline{\mathbf{E}}_c$ given by

$$
\begin{aligned}
(\blacktriangleright^c \sigma)_0 &= \mathbf{1}_{c_0} \\
(\blacktriangleright^c \sigma)_{n+1} &= c(n \leq n+1)^*(\sigma_n).
\end{aligned}
$$

Roughly:

$$\begin{tikzcd}
\sigma \arrow[d, mapsto] & \sigma_0 & \sigma_1 \arrow[l, "\sigma(0\leq1)"'] & \sigma_2 \arrow[l, "\sigma(1\leq2)"'] & {} \arrow[l, "\cdots"'] & \\
\blacktriangleright\sigma & \mathbf{1} & \sigma_0 \arrow[l, "!"'] & \sigma_1 \arrow[l, "\sigma(0\leq1)"'] & \sigma_2 \arrow[l, "\sigma(1\leq2)"'] & {} \arrow[l, "\cdots"']
\end{tikzcd}$$

- *$\blacktriangleright^c$ preserves fibred finite products*
- *$\blacktriangleright^c$ preserves all fibred limits if p is a bifibration*
- *there is a natural transformation $\mathrm{next}^c \colon \mathrm{Id} \Rightarrow \blacktriangleright^c$*

Proof Search

- Recent development with Komendantskaya and Li: Uniform proofs for coinductive predicates (coinductive Prolog)
- Instantiate the recursive proof framework for intuitionistic FOL
- Soundness over complete Herbrand models for Horn-clauses
- Soundness of uniform proofs with respect to framework
- $\Longrightarrow$ proof search for recursive proofs with proof-relevance!
- Conjecture: completeness by Bekić/compression lemma of uniform proofs
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Conclusion

Extensions and Future Directions

- Preprint: ArXiv 1802.07143 (includes extension to well-founded orders and in next version a more general soundness result)
- Next: Work out step-indexing for higher-order languages
- Next: Provide automation to obtain syntax
- Can we construct other recursive proof systems in fibrations? (Later with clocks, cyclic proof systems, . . .)
- Do we need chains? Can we use the companion instead?

Thank you for your attention!

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Diagrams are Fibred CCCs

Intuition from Kripke models

$$W, w \vDash \varphi \to \psi \quad \iff \quad \forall w \leq v.\, W, v \vDash \varphi \text{ implies } W, v \vDash \psi$$

Implication for sequences of formulas

Let $\{\varphi_n\}_{n \in \omega^{\mathrm{op}}}$ and $\{\psi_n\}_{n \in \omega^{\mathrm{op}}}$ be sequences of formulas. Define

$$(\psi \Rightarrow \varphi)_n := \bigwedge_{m \leq n} \psi_m \to \varphi_n,$$

General Exponentials

The exponential object of $\sigma, \tau \in \overline{\mathbf{E}}_c$ is given by the end

$$(\tau^\sigma)(n) = \int_{m \leq n} \big(c(m \leq n)^* \, \tau(m)\big)^{c(m \leq n)^* \, \sigma(m)}.$$

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The Löb Rule

Theorem

If $p\colon \mathbf{E} \to \mathbf{B}$ *has fibred finite limits and exponents, then also* $\overline{p}\colon \overline{\mathbf{E}} \to \overline{\mathbf{B}}$ *does.*

Notation: for $\sigma, \tau \in \overline{\mathbf{E}}_c$ *have* $\sigma^\tau \in \overline{\mathbf{E}}_c$*.*

Theorem

For every $\sigma \in \overline{\mathbf{E}}_c$ *there is a unique map in* $\overline{\mathbf{E}}_c$*, dinatural in* σ*,*

$$\mathrm{l\ddot{o}b}^c_\sigma \colon \sigma^{\blacktriangleright^c \sigma} \to \sigma.$$

Associated proof rule

$$\frac{f : \tau \times \blacktriangleright^c \sigma \to \sigma}{\mathrm{l\ddot{o}b}^c_\sigma \circ \lambda f : \tau \to \sigma}\ \textbf{(Löb)}$$

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Steps on the Final Chain

Theorem

$\overleftarrow{\Phi} = \blacktriangleright\left(\overline{\Phi}\,\overleftarrow{\Phi}\right)$, *where* $\blacktriangleright := \blacktriangleright^{K_X}$.

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For $T\colon \mathbf{E}_X \to \mathbf{E}_X$ *and* $\rho\colon T\Phi \Rightarrow \Phi T$, *there is* $\overleftarrow{\rho} : \overline{T}\,\overleftarrow{\Phi} \to \overleftarrow{\Phi}$.

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Quantifiers (Products & Coproducts)

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If for $u\colon I \to J$ in $\mathbf{B}$ the coproduct $\coprod_u\colon \mathbf{E}_I \to \mathbf{E}_J$ along u exists, then the coproduct $\coprod_{\overline{u}}\colon \overline{\mathbf{E}}_I \to \overline{\mathbf{E}}_J$ along $\overline{u}\colon K_I \to K_J$ is given by $\overline{\coprod_u}$. Similarly, the product $\prod_{\overline{u}}$ along $\overline{u}$ is given by $\overline{\prod_u}$.

Associated proof rule

Let $\pi\colon I \times J \to I$, and write $W = \overline{\pi}^*$ for weakening $W\colon \overline{\mathbf{E}}_I \to \overline{\mathbf{E}}_{I\times J}$ and $\forall_J = \prod_{\overline{\pi}}\colon \overline{\mathbf{E}}_{I\times J} \to \overline{\mathbf{E}}_I$. Then

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