Using Coalgebras to Find the Productive Among the Lazy

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Scottish Theorem Proving Seminar
07 October 2015

Outline

1. Introduction
2. Calculus for Mixed Inductive-Coinductive Definitions
3. Productivity and its Proof Principles
4. Proof Principles in Action

1 Introduction

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Introduction

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- Ensure that proofs are well-defined
- For inductive proofs: termination
- For coinductive proofs: productivity
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Proposal

Non-intrusive proof technique based on coalgebraic techniques

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Types

Definition (Types)

$$A, B ::= A + B \mid A \times B \mid A \to B \mid X \in \mathrm{TyVar} \mid \mu X.A \mid \nu X.A$$

where X occurs never on the left of $\to$, i.e., A is strictly positive.

Example

$$\begin{aligned} \mathbf{1} &:= \nu X.X \\ \mathbb{B} &:= \mathbf{1} + \mathbf{1} \\ \mathbb{B}^\omega &:= \nu X.\mathbb{B} \times X \end{aligned}$$

or more complicated mixed fixed points.

Programs I

Example (For $\mathbf{1} = \nu X.X$)

$$\langle\rangle : \mathbf{1}$$
$$\xi\,\langle\rangle = \langle\rangle$$

NB: RHS of equation must be of type X $[\mathbf{1}/X] = \mathbf{1}$.

Example (For $\mathbb{B} = \mathbf{1} + \mathbf{1}$)

$$\top, \bot : \mathbb{B} \qquad \neg : \mathbb{B} \to \mathbb{B}$$
$$\bot = \kappa_1 \langle\rangle \qquad \neg(\kappa_1\, x) = \top$$
$$\top = \kappa_2 \langle\rangle \qquad \neg(\kappa_2\, x) = \bot$$

Programs II

Example (For $\mathbb{B}^\omega = \nu X.\mathbb{B} \times X$)

- $s : B^\omega \vdash \mathsf{hd}\ s : \mathbb{B}$ with $\mathsf{hd}\, s := \pi_1(\underbrace{\xi\, s}_{(\mathbb{B}\times X)[\mathbb{B}^\omega/X]=\mathbb{B}\times B^\omega})$
- $s : B^\omega \vdash \mathsf{tl}\ s : \mathbb{B}^\omega$ with $\mathsf{tl}\, s := \pi_2(\xi\, s)$
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$$\begin{aligned} \sim : B^\omega &\to B^\omega \\ \mathsf{hd}(\sim s) &= \neg(\mathsf{hd}\, s) \\ \mathsf{tl}(\sim s) &= \sim(\mathsf{tl}\, s) \end{aligned} \qquad \begin{aligned} \mathrm{alt} &: \mathbb{B}^\omega \\ \mathsf{hd}\ \mathrm{alt} &= \bot \\ \mathsf{tl}\ \mathrm{alt} &= \sim \mathrm{alt} \end{aligned}$$

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Is alt well-defined, i.e. productive, even though not guarded?

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Reduction Behaviour

$$\mathsf{hd}(\mathsf{tl}^n\, \mathrm{alt}) \twoheadrightarrow \mathsf{hd}(\sim^n\, \mathrm{alt}) \twoheadrightarrow \neg^n(\mathsf{hd}\, \mathrm{alt})$$

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When is a program well-defined?

Slogan

A program P is well-defined iff it is terminating and the outcome of any observation we can make on P is again well-defined.

Formally

- Define coalgebra $\delta : \Lambda \to F(\Lambda)$ on programs that captures observations we can make on programs
- This gives rise to functor $S_\delta : \mathrm{Pred}_\Lambda \to \mathrm{Pred}_\Lambda$. For example

$$S_\delta(P)_{A_1 \times A_2} = \{t : A_1 \times A_2 \mid \forall i \in \{1, 2\}.\ \forall s.\ \pi_i\, t \twoheadrightarrow s \Rightarrow s \in P_{A_i}\}$$

- The set **ON** of well-defined programs is the largest set s.t.

$$\mathbf{ON} \sqsubseteq \Psi(\mathbf{ON})$$

where $\Psi(P) = \mathbf{SN} \sqcap S_\delta(\mathbf{ON})$.

Proof Principle

- This gives us obvious proof principle:

$$\frac{P \sqsubseteq \mathbf{SN} \qquad P \sqsubseteq S_\delta(P)}{P \sqsubseteq \mathbf{ON}}$$

- Difficult to work with: A predicate P that would prove $\mathrm{alt} \in \mathbf{ON}$ is necessarily infinite.
- Improve by using up-to techniques

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Lemma (Pous '07)

If T is Ψ-compatible, then

$$\frac{P \sqsubseteq \mathbf{SN} \qquad P \sqsubseteq S_\delta(T(P))}{P \sqsubseteq \mathbf{ON}}$$

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Prove Productivity of alt

Goal

Find up-to technique T, such that $P \sqsubseteq \Psi(T(P))$ with

$$P_{\mathbb{B}^\omega} = \{\mathrm{alt}\}$$
$$P_A = \emptyset, \qquad A \neq \mathbb{B}^\omega$$

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1. Define $T_1(P) = \mathbf{ON} \sqcup P$.
2. Define T_2 s.t. $(\sim s) \in T_2(P)_{\mathbb{B}^\omega}$ for all $s \in P_{\mathbb{B}^\omega}$.
3. Define T_3 s.t. $(\mathrm{hd}\, f) \in T_3(P)_{\mathbb{B}^\omega}$, if $e \in P_{\mathbb{B}^\omega}$ for $\mathrm{hd}\, f = e$ in the program, and same for $\mathrm{tl}\, f$.
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Lemma (Pous '07)

If F, G are Ψ-compatible, then so is $G \circ F$.

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Proof that $\mathrm{alt} \in \mathbf{ON}$

- Recall: $\mathrm{alt} : \mathbb{B}^\omega$ $\quad$ $\mathsf{hd}\ \mathrm{alt} = \bot$ $\quad$ $\mathsf{tl}\ \mathrm{alt} = \sim \mathrm{alt}$
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