Models of Inductive-Coinductive Logic Programs

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Inductive Logic Programs

- Classically logic programs generate finite terms
- Proof is finite sequence of forward clause applications
- Canonical model: Least Herbrand model
- SLD resolution as semi-decision procedure

Example (Natural Numbers)

Consider

$$\begin{aligned} 1: &\quad \mathrm{nat}(0) \longleftarrow \\ 2: &\quad \mathrm{nat}(\mathrm{s}(x)) \longleftarrow \mathrm{nat}(x) \end{aligned}$$

We get

- $\mathrm{nat}(0)$ by 1
- $\mathrm{nat}(\mathrm{s}(0))$ by 1 and then 2
- $\dots$

Thereby generating the natural numbers

Coinductive Logic Programs

- Later logic programs have been extended to possibly infinite terms
- Proof is a possibly infinite sequence of backward clause applications
- Canonical model: Largest Herbrand model
- SLD resolution

Example (Streams)

Consider

$$3: \quad \mathrm{str}(\mathrm{cons}(x, y)) \longleftarrow \mathrm{nat}(x), \mathrm{str}(y)$$

Let $o = \mathrm{cons}(0, \mathrm{cons}(0, \ldots)) = \mathrm{cons}(0, o)$.

- Check $\mathrm{str}(o)$
- By 3, we need to check that $\mathrm{nat}(0)$ and $\mathrm{str}(o)$
- $\mathrm{nat}(0)$ by 1
- Repeat indefinitely (or use some mechanism to close the proof here)

A Problem

- In coinductive interpretation of combined program we have

$$\mathrm{nat}(\mathrm{s}^{\omega})$$

- Thus, we generate natural numbers with an extra element at infinity
- This leads to further spurious elements:

$$\mathrm{str}(\mathrm{cons}(\mathrm{s}^{\omega}, o))$$

Solution

Select for each predicate locally whether to interpret it inductively or coinductively.

Inductive-Coinductive Logic Programs I

- Specify for each predicate whether its rules may be applied infinitely often in succession
- Proof is a sequence of forward and backward clause applications
- Canonical model: Mixed fixed point model
- Future: resolution as semi-decision procedure

Inductive-Coinductive Logic Programs II

Example (Streams over natural numbers)

Consider program annotated with parities

$$\mathsf{par}(\mathrm{nat}) = \mu \qquad \mathsf{par}(\mathrm{str}) = \nu$$

$$1: \quad \mathrm{nat}(0) \longleftarrow$$

$$2: \quad \mathrm{nat}(\mathrm{s}(x)) \longleftarrow \mathrm{nat}(x)$$

$$3: \quad \mathrm{str}(\mathrm{cons}(x, y)) \longleftarrow \mathrm{nat}(x), \mathrm{str}(y)$$

We get

- As before: $\mathrm{nat}(0)$, $\mathrm{nat}(\mathrm{s}(0))$, $\ldots$
- And: $\mathrm{str}(o)$ for $o = \mathrm{cons}(0, o)$

Forbidden

- s^ω requires infinite consecutive applications of rule 2

Use of Inductive-Coinductive Logic Programs I

Example (Substream Relation)

Extend program with

$$\mathsf{par}(\mathrm{sub}) = \nu \qquad \mathsf{par}(\mathrm{sub}_\mu) = \mu$$

$$4: \quad \mathrm{sub}(x, y) \longleftarrow \mathrm{sub}_\mu(x, y)$$

$$5: \quad \mathrm{sub}_\mu(\mathrm{cons}(n, x), \mathrm{cons}(n, y)) \longleftarrow \mathrm{nat}(n), \mathrm{sub}(x, y)$$

$$6: \quad \mathrm{sub}_\mu(x, \mathrm{cons}(n, y)) \longleftarrow \mathrm{nat}(n), \mathrm{sub}_\mu(x, y)$$

Use of Inductive-Coinductive Logic Programs II

Example

Define

$$o = \mathrm{cons}(0, o)$$

$$\mathrm{alt} = \mathrm{cons}(0, \mathrm{cons}(1, \mathrm{alt}))$$

Then $\mathrm{sub}(o, \mathrm{alt})$

$$\mathsf{par}(\mathrm{sub}) = \nu \qquad \mathsf{par}(\mathrm{sub}_\mu) = \mu$$

$$4: \quad \mathrm{sub}(x, y) \longleftarrow \mathrm{sub}_\mu(x, y)$$

$$5: \quad \mathrm{sub}_\mu(\mathrm{cons}(n, x), \mathrm{cons}(n, y)) \longleftarrow \mathrm{nat}(n), \mathrm{sub}(x, y)$$

$$6: \quad \mathrm{sub}_\mu(x, \mathrm{cons}(n, y)) \longleftarrow \mathrm{nat}(n), \mathrm{sub}_\mu(x, y)$$

$$\mathrm{sub}(o, \mathrm{alt}) \overset{(4)}{\longleftarrow} \mathrm{sub}_\mu(o, \mathrm{alt})$$

$$\overset{(5)}{\longleftarrow} \mathrm{sub}(o, \mathrm{cons}(1, \mathrm{alt})) \overset{(4)}{\longleftarrow} \mathrm{sub}_\mu(o, \mathrm{cons}(1, \mathrm{alt})) \overset{(6)}{\longleftarrow} \mathrm{sub}_\mu(o, \mathrm{alt})$$

$$\overset{(5)}{\longleftarrow} \mathrm{sub}(o, \mathrm{cons}(1, \mathrm{alt})) \overset{(4)}{\longleftarrow} \mathrm{sub}_\mu(o, \mathrm{cons}(1, \mathrm{alt})) \overset{(6)}{\longleftarrow} \mathrm{sub}_\mu(o, \mathrm{alt})$$

$$\overset{(5)}{\longleftarrow} \quad \ldots$$

Use of Inductive-Coinductive Logic Programs III

Inductive nature of sub_μ

Derivations of sub_μ are interleaved by derivations for sub.

$$
\begin{aligned}
&\mathrm{sub}(o, \mathrm{alt}) \xleftarrow{(4)} \mathrm{sub}_\mu(o, \mathrm{alt}) \xleftarrow{(5)} \mathrm{sub}(o, \mathrm{cons}(1, \mathrm{alt}))\\
&\xleftarrow{(4)} \mathrm{sub}_\mu(o, \mathrm{cons}(1, \mathrm{alt})) \xleftarrow{(6)} \mathrm{sub}_\mu(o, \mathrm{alt}) \xleftarrow{(5)} \mathrm{sub}(o, \mathrm{cons}(1, \mathrm{alt}))\\
&\xleftarrow{(4)} \mathrm{sub}_\mu(o, \mathrm{cons}(1, \mathrm{alt})) \xleftarrow{(6)} \mathrm{sub}_\mu(o, \mathrm{alt}) \xleftarrow{(5)} \mathrm{sub}(o, \mathrm{cons}(1, \mathrm{alt}))\\
&\xleftarrow{(4)} \qquad \ldots
\end{aligned}
$$

Inductive-Coinductive Logic Programs

- Σ — Term signature
- Δ — Relation signature
- $\Sigma^*(V)$ — Finite terms over variables in V
- Σ^∞ — Potentially infinite ground terms
- $\varphi = Q(\vec{t})$ — Formula with $Q \in \Delta$, $\vec{t} = (t_1, \ldots, t_{\mathrm{ar}\,Q})$, $t_i \in \Sigma^*(V)$
- $\varphi \longleftarrow S$ — Horn clause with φ formula and S set of formulas
- (Φ, par) — Logic program with Φ set of clauses and $\mathrm{par} \colon \Delta \to \{\mu, \nu\}$

Clauses as Relation Transformer

$$\widehat{\Phi}\colon \prod_{Q \in \Delta} \mathrm{Rel}_{\mathsf{ar}\,Q}(\Sigma^\infty) \to \prod_{Q \in \Delta} \mathrm{Rel}_{\mathsf{ar}\,Q}(\Sigma^\infty)$$

$$\widehat{\Phi}(F)(Q) := \bigcup_{Q(\vec{t}) \longleftarrow S \,\in\, \Phi} \Big\{ \vec{t}[\sigma] \Big| \sigma\colon V \to \Sigma^\infty \text{ and } \forall P(\vec{s}) \in S.\ \vec{s}[\sigma] \in F(P) \Big\},$$

Components for each parity $\rho \in \{\mu, \nu\}$

$$\widehat{\Phi}_\rho : \prod_{Q \in \Delta} \mathrm{Rel}_{\mathsf{ar}\,Q}(\Sigma^\infty) \to \prod_{Q \in \Delta_\rho} \mathrm{Rel}_{\mathsf{ar}\,Q}(\Sigma^\infty)$$

$$\widehat{\Phi}_\rho(Q) := \widehat{\Phi}(Q)|_{\Delta_\rho}$$

Models of Inductive-Coinductive LPs

Definition

A Φ-*model* $\mathcal{M}$ is given by an $\mathcal{M} \in \prod_{Q \in \Delta} \mathrm{Rel}_{\mathsf{ar}\, Q}(\Sigma^{\infty})$, such that

$$\widehat{\Phi}_{\mu}(\mathcal{M}) \sqsubseteq \mathcal{M}_{\mu} \quad \text{and} \quad \mathcal{M}_{\nu} \sqsubseteq \widehat{\Phi}_{\nu}(\mathcal{M}),$$

where $\sqsubseteq$ is point-wise inclusion and $\mathcal{M}_{\rho}$ is the restriction $\mathcal{M}|_{\Delta_{\rho}}$.

Example Models I

Example (Natural Numbers)

- With $\text{par}(\mathrm{nat}) = \mu$, we have an expected model:

$$\mathcal{M}(N) = \{\mathrm{s}^n(0) \mid n \in \mathbb{N}\} = \Sigma_N^*.$$

- Non-standard model is also possible:

$$\mathcal{M}_\infty(N) = \mathcal{M}(N) \cup \{\mathrm{s}^\omega\} = \Sigma_N^\infty$$

- Smaller models are not possible:

$$\mathcal{M}_{\leq 1}(N) = \{0, \mathrm{s}(0)\}$$

  Because

$$\widehat{\Phi}(\mathcal{M}_{\leq 1})(N) = \{0, \mathrm{s}(0), \mathrm{s}^2(0)\} \not\subseteq \mathcal{M}_{\leq 1}(N).$$

Example Models II

Example (Extended Natural Numbers)

- With $\text{par}(\mathrm{nat}) = \nu$, all of these are models
- Non-example:

$$\mathcal{M}_1 = \{\mathrm{s}(0)\},$$

because $\mathcal{M}_1$ is not backwards closed:

$$\mathcal{M}_1 \not\subseteq \widehat{\Phi}(\mathcal{M}_1)(N) = \{0, \mathrm{s}^2(0)\}.$$

Characterisation of models

Definition

Satisfaction relative Φ-model $\mathcal{M}$:

$$\mathcal{M} \vDash Q(\vec{t}) := \vec{t} \in \mathcal{M}(Q).$$

Satisfaction of sentences:

$$\mathcal{M} \vDash S := \forall \phi \in S.\ \mathcal{M} \vDash \phi$$

Theorem

Let $\mathcal{M}$ be Φ-model. For all $Q \in \Delta_\mu$ we have (forward closure)

$$\forall (Q(\vec{t}) \longleftarrow S) \in \Phi.\ \forall \sigma.\ \mathcal{M} \vDash S[\sigma] \ \Rightarrow\ \mathcal{M} \vDash Q\left(\vec{t}[\sigma]\right).$$

Dually, for all $Q \in \Delta_\nu$, we have (backward closure)

$$\mathcal{M} \vDash Q(\vec{u}) \ \Rightarrow\ \exists (Q(\vec{t}) \longleftarrow S) \in \Phi.\ \exists \sigma.\ \vec{u} = \vec{t}[\sigma] \wedge \mathcal{M} \vDash S[\sigma].$$

What Now?

- We can compute fixed point model:

$$\nu X.\mu Y.\widehat{\Phi}(X, Y)$$

- Extend SLD-resolution – also category theoretical perspective
- Fixed point model serves as target for resolution (soundness and possibly completeness)

Thank you very much for your attention!