### Breaking the Loop Recursive Proofs for Coinductive Predicates

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# **Original Motivation**

- Syntactic logic for program equivalence in my thesis
- Recursive proof system based on later modality
- Recursion gives rise to proof search
- Many of the constructions are pedestrian
- Need for an abstract framework

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### Motivation

# Stream Differential Equations

Example (Constant Streams)

$$a^{\omega}: \mathbb{R}^{\omega} \qquad a_0^{\omega} = a \qquad (a^{\omega})' = a^{\omega}$$

Example (Point-wise Stream Addition)

Example (Stream of Positive Numbers)

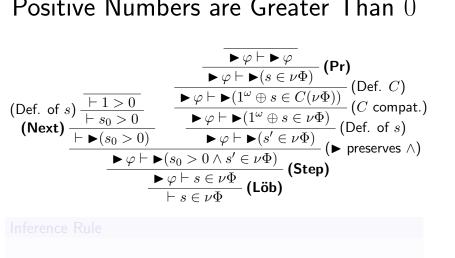
$$s: \mathbb{R}^{\omega}$$
  $s_0 = 1$   $s' = 1^{\omega} \oplus s$ 

### Point-wise Positive Streams

Example (Predicate Transformer)

$$\Phi(P \subseteq \mathbb{R}^{\omega}) = \{ s \in \mathbb{R}^{\omega} \mid s_0 > 0 \land s' \in P \}$$

- $\Phi$  monotone
- Greatest fixed point  $u\Phi$  exists
- $s \in \nu \Phi$  iff s is point-wise greater than 0





$$(\operatorname{Def. of} s) \xrightarrow[\vdash 1 > 0]{} \frac{(1 + 1 > 0)}{(1 + s_0 > 0)} = (1 + 1) + (1 +$$

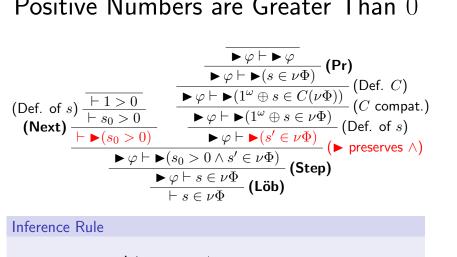
Inference Rule  

$$\begin{split} \varphi &:= s \in \nu \Phi \\ \frac{\Delta, \blacktriangleright \varphi \vdash \varphi}{\Delta \vdash \varphi} \text{ (Löb)} \end{split}$$

$$(\operatorname{Def. of} s) \xrightarrow[\vdash 1 > 0]{} \frac{(1 + 1 > 0)}{(1 + s_0 > 0)} = (1 + 1) + (1 +$$

Inference Rule

$$\frac{\Delta \vdash \blacktriangleright (s \in \Phi(\nu \Phi))}{\Delta \vdash s \in \nu \Phi}$$
(Step)  
$$\Phi(P) = \{s \in \mathbb{R}^{\omega} \mid s_0 > 0 \land s' \in P\}$$



 $\frac{\Delta \vdash \blacktriangleright \varphi \land \blacktriangleright \psi}{\Delta \vdash \blacktriangleright (\varphi \land \psi)} (\blacktriangleright \text{ preserves } \land)$ 

$$(\operatorname{Def. of} s) \underbrace{\frac{\vdash 1 > 0}{\vdash s_0 > 0}}_{(\operatorname{Next})} \underbrace{\frac{\vdash 1 > 0}{\vdash (s_0 > 0)}}_{\vdash (s_0 > 0)} \underbrace{\frac{ \vdash \varphi \vdash (1^{\omega} \oplus s \in C(\nu\Phi))}{\vdash \varphi \vdash (1^{\omega} \oplus s \in V\Phi)}}_{(\operatorname{Pr})} (\operatorname{Def. } C)$$

$$(C \text{ compat.})$$

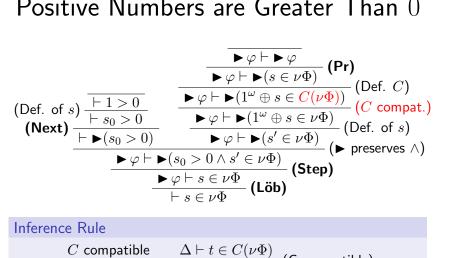
$$\underbrace{\frac{ \vdash \varphi \vdash (s_0 > 0)}{\vdash \varphi \vdash (s_0 > 0 \land s' \in \nu\Phi)}}_{\downarrow \varphi \vdash (s_0 > 0 \land s' \in \nu\Phi)} (\operatorname{Def. of} s)$$

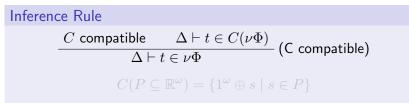
$$\underbrace{\frac{ \vdash \varphi \vdash (s_0 > 0 \land s' \in \nu\Phi)}{\vdash s \in \nu\Phi}}_{\vdash s \in \nu\Phi} (\operatorname{L\ddot{o}b})$$

Inference Rule $\frac{\Delta \vdash \varphi}{\Delta \vdash \blacktriangleright \varphi} \text{ (Next)}$ 

$$(\operatorname{Def. of} s) \frac{\overline{\vdash 1 > 0}}{\vdash s_0 > 0} \qquad \frac{\varphi \vdash \varphi}{\vdash \varphi \vdash (1^{\omega} \oplus s \in \nu\Phi)} (\operatorname{Pr}) (\operatorname{Def. } C) \\ \xrightarrow{\varphi \vdash \varphi \vdash (1^{\omega} \oplus s \in \nu\Phi)} (\varphi \vdash (1^{\omega} \oplus s \in \nu\Phi)) (C \text{ compat.})} \\ \xrightarrow{\varphi \vdash \varphi \vdash (s_0 > 0)} \xrightarrow{\varphi \vdash \varphi \vdash (1^{\omega} \oplus s \in \nu\Phi)} (\varphi \vdash (s_0 = \nu\Phi)) (\varphi \vdash (s' \in \nu\Phi)) (\varphi \vdash (s' \in \nu\Phi))} \\ \xrightarrow{\varphi \vdash s \in \nu\Phi} (\varphi \vdash s \in \nu\Phi) (\varphi \vdash s \in \nu\Phi)} (\varphi \vdash s \in \nu\Phi) (\varphi \vdash s \in \nu\Phi)} (\varphi \vdash s \in \nu\Phi) (\varphi \vdash s \in \mu \Phi) (\varphi \vdash s \in \mu\Phi) (\varphi \vdash s \vdash x \to \mu\Phi) (\varphi \vdash s \vdash x \to \mu\Phi) (\varphi \vdash s \vdash x \to \mu\Phi) (\varphi \vdash x \vdash x \to \mu\Phi) (\varphi \vdash x \to \mu\Phi)$$

Inference Rule $s: \mathbb{R}^{\omega} \qquad s_0 = 1 \qquad s' = 1^{\omega} \oplus s$ 



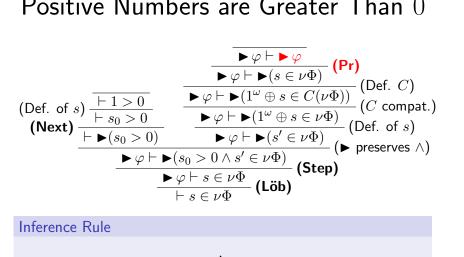


$$(\operatorname{Def. of} s) \xrightarrow[\vdash 1 > 0]{} \frac{(1 + 1 > 0)}{(1 + s_0 > 0)} = (1 + 1) \xrightarrow[\vdash \infty]{} \frac{\varphi \vdash (1^{\omega} \oplus s \in V\Phi)}{(1^{\omega} \oplus s \in C(\nu\Phi))} \xrightarrow[\vdash \infty]{} \frac{\varphi \vdash (1^{\omega} \oplus s \in V\Phi)}{(1^{\omega} \oplus s \in \nu\Phi)} \xrightarrow[\vdash \infty]{} (C \text{ compat.}) \xrightarrow[\vdash \infty]{} \frac{\varphi \vdash (1^{\omega} \oplus s \in \nu\Phi)}{(1^{\omega} \oplus s \in \nu\Phi)} \xrightarrow[\vdash \infty]{} (Pr) \xrightarrow[\vdash \infty]{} (C \text{ compat.}) \xrightarrow[\vdash \infty]{} (C \text{ compat.}) \xrightarrow[\vdash \infty]{} \frac{\varphi \vdash (1^{\omega} \oplus s \in \nu\Phi)}{(1^{\omega} \oplus s \in \nu\Phi)} \xrightarrow[\vdash \infty]{} (C \text{ compat.}) \xrightarrow[\vdash \infty$$

Inference Rule  

$$\frac{C \text{ compatible } \Delta \vdash t \in C(\nu\Phi)}{\Delta \vdash t \in \nu\Phi} \text{ (C compatible)}$$

$$C(P \subseteq \mathbb{R}^{\omega}) = \{1^{\omega} \oplus s \mid s \in P\}$$



$$\frac{\varphi \in \Delta}{\Delta \vdash \varphi} (\mathsf{Pr})$$

### Idea

# Extending a Logic

- Given a logic  ${\mathcal L}$  with formulas  $\varphi$  and provability  $\Gamma \mid \Delta \vdash \varphi$
- Construct a new logic  $\overline{\mathcal{L}}$  with the same propositional and first-order connectives, . . .
- ... and a new connective ►, the later modality, that fulfils the axioms for the later modality ...
- ... and enables coinductive predicates and up-to techniques

Rules for the later modality

$$\frac{\Gamma \mid \Delta \vdash \varphi}{\Gamma \mid \Delta \vdash \blacktriangleright \varphi} \text{ (Next) } \frac{\Gamma \mid \Delta \vdash \blacktriangleright (\varphi \rightarrow \psi)}{\Gamma \mid \Delta \vdash \blacktriangleright \varphi \rightarrow \blacktriangleright \psi} \text{ (Mon)}$$

$$\frac{\Gamma \mid \Delta, \blacktriangleright \varphi \vdash \varphi}{\Gamma \mid \Delta \vdash \varphi} \text{ (Löb)}$$

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- ... and enables coinductive predicates and up-to techniques

Rules for coinductive predicates and up-to techniques

$$\frac{\Gamma \mid \Delta \vdash \blacktriangleright (s \in \Phi(\nu \Phi))}{\Gamma \mid \Delta \vdash s \in \nu \Phi}$$
(Step)

 $\frac{C \text{ is } \Phi \text{-compatible } \Gamma \mid \Delta \vdash t \in C(\nu \Phi)}{\Gamma \mid \Delta \vdash t \in \nu \Phi} \text{ (Up-to)}$ 

### Setup

# Fibrations

- Fibrations provide abstraction of first- (and higher-)order logic
- $\mathbf{B}$  Category of typed contexts and terms
- ${f E}$  Category of formulas with variables typed in  ${f B}$
- $p \colon \mathbf{E} \to \mathbf{B}$  functor that assigns to a formula its context

#### Example

- Set-based predicates:  $Pred \rightarrow Set$
- Quantitative predicates:  $qPred \rightarrow \mathbf{Set}$
- Syntactic logic over syntactic terms:  $\mathcal{L} 
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- Set-indexed families (dependent types):  $Fam(\mathbf{C}) \rightarrow \mathbf{Set}$

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# Example: Quantitative Predicates

### Category of quantitative predicates

 $\mathbf{qPred} = \begin{cases} \mathsf{objects:} & (X,\delta) \text{ with } X \in \mathbf{Set} \text{ and } \delta \colon X \to [0,1] \\ \mathsf{morphisms:} & f \colon (X,\delta) \to (Y,\gamma) \text{ if } f \colon X \to Y \text{ in } \mathbf{Set} \\ & \mathsf{and} \ \delta \leq \gamma \circ f \end{cases}$ 

Reindexing along  $u: X \to Y$  gives fibration  $\mathbf{qPred} \to \mathbf{Set}$  $u^*(Y, \gamma) = (X, \lambda x, \gamma(u(x)))$ 

#### Products and Exponents

$$\begin{split} (\delta\times\gamma)(x) &= \min\{\delta(x),\gamma(x)\}\\ \Bigl(\gamma^\delta\Bigr)(x) &= \begin{cases} 1, & \delta(x) \leq \gamma(x)\\ \gamma(x), & \text{otherwise} \end{cases} \end{split}$$

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### **Coinductive Predicates**

Predicate lifting G of behaviour functor F

commutes and G preserves Cartesian morphisms.

Predicate transformer for coalgebra  $c \colon X \to FX$  $\Phi := \mathbf{E}_X \xrightarrow{G} \mathbf{E}_{FX} \xrightarrow{c^*} \mathbf{E}_X$ 

#### Coinductive predicate

Final coalgebra  $\xi \colon \nu \Phi \to \Phi(\nu \Phi)$  for  $\Phi$ 

# $\omega^{\mathrm{op}}\text{-}\mathsf{Diagrams}$ in Fibrations

Category of Descending Chains

$$\overline{\mathbf{C}} = [\omega^{\mathrm{op}}, \mathbf{C}] =$$
 "category of functors  $\omega^{\mathrm{op}} \to \mathbf{C}$ "

#### **Constant-Index Chains**

$$\overline{\mathbf{E}}_X := \overline{\mathbf{E}}_{K_X} \cong \overline{\mathbf{E}}_X$$
  
If  $\sigma \in \overline{\mathbf{E}}_X$ , then  $p(\sigma_n) = \overline{p}(\sigma)_n = (K_X)_n = X$ .

The final chain  $\overleftarrow{\Phi}\in \overline{\mathbf{E}}_X$ 

$$\overleftarrow{\Phi} := \mathbf{1} \xleftarrow{!} \Phi(\mathbf{1}) \xleftarrow{\Phi(!)} \Phi^2(\mathbf{1}) \xleftarrow{\Phi^2(!)} \Phi^3(\mathbf{1}) \xleftarrow{\cdots}$$

If  $\Phi$  preserves  $\omega^{\text{op}}$ -limits, then maps  $A \to \nu \Phi$  in  $\mathbf{E}_X$  can be given by maps  $K_A \to \overleftarrow{\Phi}$  in  $\overline{\mathbf{E}}_X$ .

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# Greater-Than-0 Example

Example (Predicate lifting and coinductive predicate)

 $F: \mathbf{Set} \to \mathbf{Set} \quad G: \operatorname{Pred} \to \operatorname{Pred}$  $F = \mathbb{R} \times \operatorname{Id} \qquad G(X, P) = (FX, \{(a, x) \mid a > 0 \land x \in P\})$ 

Predicate transformer

 $\Phi = \langle \mathrm{hd}, \mathrm{tl} \rangle^* \circ G$ 

Coinductive predicate

 $\nu\Phi\subseteq\Phi(\nu\Phi)$ 

### Example (Notation)

Given a descending chain  $\sigma \in \overline{\operatorname{Pred}}_X$ , we define  $\vdash \sigma := \overline{\mathbf{1}}_X \sqsubseteq \sigma$  ( $\iff$  there exists  $\overline{\mathbf{1}}_X \to \sigma$ )  $x \in \sigma := \sigma^{K_{\{x\}}}$ 

 $\vdash s \in \overleftarrow{\Phi} \iff \forall n \in \mathbb{N}. \, s \in \overleftarrow{\Phi}_n \stackrel{Thm}{\iff} s \in \nu \Phi \iff s \text{ greater t. 0}$ 

# Greater-Than-0 Example

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 $\begin{array}{ll} F \colon \mathbf{Set} \to \mathbf{Set} & G \colon \mathrm{Pred} \to \mathrm{Pred} \\ F = \mathbb{R} \times \mathrm{Id} & G(X, P) = (FX, \{(a, x) \mid a > 0 \land x \in P\}) \end{array}$ 

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 $\vdash s \ensuremath{\overline{\in}} \ensuremath{\overleftarrow{\Phi}} \ensuremath{\longleftrightarrow} \ensuremath{\vartheta} n \ensuremath{\in} \mathbb{N}. s \ensuremath{\in} \ensuremath{\overleftarrow{\Phi}} \ensuremath{n} \ensuremath{\overleftarrow{\Phi}} \ensuremath{a} \ensuremath{\varepsilon} \ensuremath{\omega} \ensuremath{\omega} \ensuremath{\omega} \ensuremath{s} \ensuremath{s} \ensuremath{\varepsilon} \ensuremath{\omega} \ensuremath{a} \ensurem$ 

# Later Modality

### Theorem

For each  $c \in \overline{\mathbf{B}}$ , there is a fibred functor  $\mathbf{P}^c \colon \overline{\mathbf{E}}_c \to \overline{\mathbf{E}}_c$ .

- ▶<sup>c</sup> preserves fibred finite products
- $\triangleright^c$  preserves all fibred limits if p is a bifibration
- there is a natural transformation next<sup>c</sup>: Id ⇒ ▶<sup>c</sup>

$$\frac{f: \tau \to \sigma}{\blacktriangleright f: \blacktriangleright \tau \to \blacktriangleright \sigma}$$
(Mon) 
$$\frac{f: \tau \to \sigma}{\operatorname{next}^c \circ f: \tau \to \blacktriangleright^c \sigma}$$
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# The Löb Rule

#### Theorem

If  $p: \mathbf{E} \to \mathbf{B}$  has fibred finite limits and exponents, then also  $\overline{p}: \overline{\mathbf{E}} \to \overline{\mathbf{B}}$  does. Notation: for  $\sigma, \tau \in \overline{\mathbf{E}}_c$  have  $\sigma^{\tau} \in \overline{\mathbf{E}}_c$ .

### Theorem

For every  $\sigma \in \overline{\mathbf{E}}_c$  there is a unique map in  $\overline{\mathbf{E}}_c$ , dinatural in  $\sigma$ ,

$$\operatorname{l\"ob}_{\sigma}^{c} \colon \sigma^{\blacktriangleright^{c} \sigma} \to \sigma.$$

$$\frac{f \colon \tau \times \blacktriangleright^c \sigma \to \sigma}{ \text{löb}_{\sigma}^c \circ \lambda f \colon \tau \to \sigma} (\textbf{Löb})$$

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### Steps on the Final Chain

Theorem  

$$\overleftarrow{\Phi} = \blacktriangleright (\overline{\Phi} \overleftarrow{\Phi}), \text{ where } \blacktriangleright := \blacktriangleright^{K_X}.$$

$$\frac{f\colon\tau\to\mathbf{P}\left(\overline{\Phi}\overleftarrow{\Phi}\right)}{f\colon\tau\to\overleftarrow{\Phi}}$$
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# Up-To Techniques

Theorem

For  $T : \mathbf{E}_X \to \mathbf{E}_X$  and  $\rho : T\Phi \Rightarrow \Phi T$ , there is  $\overleftarrow{\rho} : \overline{T}\overleftarrow{\Phi} \to \overleftarrow{\Phi}$ .

$$\frac{f: \tau \to \overline{T} \overleftarrow{\Phi} \qquad \rho: T\Phi \Rightarrow \Phi T \ (T \text{ compatible})}{\overleftarrow{\rho} \circ f: \tau \to \overleftarrow{\Phi}} \ (\text{Up-to})$$

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# Quantifiers (Products & Coproducts)

#### Theorem

If for  $u: I \to J$  in **B** the coproduct  $\coprod_u: \mathbf{E}_I \to \mathbf{E}_J$  along u exists, then the coproduct  $\coprod_{\overline{u}}: \overline{\mathbf{E}}_I \to \overline{\mathbf{E}}_J$  along  $\overline{u}: K_I \to K_J$  is given by  $\fbox{\Pi}_u$ . Similarly, the product  $\prod_{\overline{u}}$  along  $\overline{u}$  is given by  $\fbox{\Pi}_u$ .

### Associated proof rule

Let  $\pi: I \times J \to I$ , and write  $W = \overline{\pi}^*$  for weakening  $W: \overline{\mathbf{E}}_I \to \overline{\mathbf{E}}_{I \times J}$  and  $\forall_J = \prod_{\overline{\pi}}: \overline{\mathbf{E}}_{I \times J} \to \overline{\mathbf{E}}_I$ . Then

$$f: W\tau \longrightarrow \sigma$$

$$\check{f}: \quad \tau \longrightarrow \forall_J \sigma$$

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$$\check{f}: \qquad \tau \longrightarrow \forall_J \sigma$$

### Conclusion

# **Related Systems**

- Parameterised coinduction only for lattices; works on fixed points
- CIRC cyclic proof system for coinductive predicates; hard to understand and hand-crafted
- Cyclic proof systems purely syntactic (??), hence have to be hand-crafted; rely on global correctness conditions
- (Bisimulation) Games also rely on global parity conditions; proof steps in presented system can be seen as challenge-response pairs
- Step-indexed relations instance of this and the framework by Birkedal et al.

## **Extensions and Future Directions**

- Preprint: ArXiv 1802.07143
- Publication with more examples etc. under review
- Extend to larger ordinals; the CCC result is already general, the results about the final chain need work:

$$(\blacktriangleright \sigma)_{\alpha} = \lim_{\beta < \alpha} \sigma_{\beta}$$

- Properly apply to motivating, syntactic example; possibly by automatically extracting a syntactic logic
- Can we construct other recursive proof systems in fibrations? (Later with clocks, cyclic proof systems, ...)

Thank you very much for your attention!

# Diagrams are Fibred CCCs

Intuition from Kripke models

 $W,w\vDash\varphi\rightarrow\psi\quad\iff\quad\forall w\le v.\,W,v\vDash\varphi\text{ implies }W,v\vDash\psi$ 

Implication for sequences of formulas

Let  $\{\varphi_n\}_{n\in\omega^{\mathrm{op}}}$  and  $\{\psi_n\}_{n\in\omega^{\mathrm{op}}}$  be sequences of formulas. Define

$$(\psi \Rightarrow \varphi)_n := \bigwedge_{m \le n} \psi_m \to \varphi_n,$$

#### General Exponentials

The exponential object of  $\sigma, \tau \in \overline{\mathbf{E}}_c$  is given by the end

$$(\tau^{\sigma})(n) = \int_{m \le n} \left( c(m \le n)^* \tau(m) \right)^{c(m \le n)^* \sigma(m)}$$

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$$(\psi \Rightarrow \varphi)_n := \bigwedge_{m \le n} \psi_m \to \varphi_n,$$

#### General Exponentials

The exponential object of  $\sigma, \tau \in \overline{\mathbf{E}}_c$  is given by the end

$$(\tau^{\sigma})(n) = \int_{m \le n} \left( c(m \le n)^* \tau(m) \right)^{c(m \le n)^* \sigma(m)}$$

# Diagrams are Fibred CCCs

Intuition from Kripke models

 $W,w\vDash\varphi\rightarrow\psi\quad\iff\quad\forall w\le v.\,W,v\vDash\varphi\text{ implies }W,v\vDash\psi$ 

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### Recursive Logic

## Later Modality

#### Theorem

For each  $c \in \overline{\mathbf{B}}$ , there is a fibred functor  $\mathbf{P}^c \colon \overline{\mathbf{E}}_c \to \overline{\mathbf{E}}_c$  given by

$$(\blacktriangleright^c \sigma)_0 = \mathbf{1}_{c_0}$$
$$(\blacktriangleright^c \sigma)_{n+1} = c(n \le n+1)^* (\sigma_n).$$

- ►<sup>c</sup> preserves fibred finite products
- ►<sup>c</sup> preserves all fibred limits if p is a bifibration
- there is a natural transformation  $next^c$ :  $Id \Rightarrow \triangleright^c$

### Associated proof rules

$$\frac{f:\tau \to (\blacktriangleright^c \sigma) \times (\blacktriangleright^c \sigma')}{\check{f}:\tau \to \blacktriangleright^c (\sigma \times \sigma')} \quad \frac{f:\tau \to \sigma}{\operatorname{next}^c \circ f:\tau \to \blacktriangleright^c \sigma}$$

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