# Composition and Recursion for Causal Structures

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**Abstract**

Causality appears in various contexts as a property where present behaviour can only depend on past events, but not on future events. In this paper, we compare three different notions of causality that capture the idea of causality in the form of restrictions on morphisms between coinductively defined structures, such as final coalgebras and chains, in fairly general categories. We then focus on one presentation and show that it gives rise to a traced symmetric monoidal category of causal morphisms. This shows that causal morphisms are closed under sequential and parallel composition and, crucially, under recursion.

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## 1 Introduction

Causality appears in various fields of science as the property that the output of a system at given time only depends on past and present inputs. This is particularly well-understood for computations on streams and various approaches to define causal maps on streams have been proposed [7]. More generally, distributive laws have been identified to give rise, and in the category of sets also coincide with, causal maps [14]. Such distributive laws provide a very neat formalism for constructing simultaneously several causal maps but are notoriously difficult to use in compositional specifications [5]. Our aim here is to provide a compositional framework for causal maps, in which such maps can be constructed by sequential composition, parallel composition and recursion. This framework is built around the idea of graphical calculi that arise from traced monoidal categories that allow us to construct and reason about morphisms with string diagrams.

The first question that arises is what causal maps are in general. A robust definition can be given by considering maps on final coalgebras. Suppose that $F$ is a functor on some category $\mathbf{C}$ and that it has a final coalgebra with carrier $\nu F$, which arises as the limit of a sequence of approximations that we denote by $\Phi F$. The final coalgebra $\nu F$ comes with projections $p_i \colon \nu F \to (\Phi F)_i$ that allow us to inspect an element in $\nu F$ up to stage $i$ of the approximation. Intuitively, a map $f \colon \nu F \to \nu F$ is causal if the $i$th approximation of its output only depends on the $i$th approximation of the input. This notion has been formalised by Rot and Pous [14] and we recap the formal definition in Section 3. For the purpose of this introduction, it suffices to say that one can show that causal maps can equivalently be represented by chain maps $\Phi F \to \Phi F$, which are families of maps for every approximation stage that are consistent across approximation stages. Formally, one considers $\Phi F$ as a diagram in $\mathbf{C}$ and a chain map is then a natural transformation.

Thus, there are two equivalent ways of approaching causality. Why would we choose one over the other? Causal maps on final coalgebras have the advantage that they are easy to understand and calculate. However, to attain our goal of compositional reasoning for causal maps, it is better to let go of these for a moment and work with chain maps instead. This


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gives us access to powerful tools for recursion that is akin to that of domain theory [4, 3]. Using these tools and some ideas from monoidal categories, we will be able to draw diagrams such as those in Figure 1.

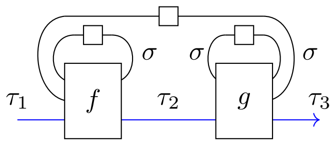



**Figure 1** Circuit with feedback loops and parameters

The interpretation of Figure 1 is that $f$ and $g$ are two causal maps that connected in various ways, including recursive feedback loops. Each of the maps has a small feedback loop and then they are tied together in one big loop. On the loops are small boxes that can be seen as registers that store information in between computation steps. It should be noted that this is an analogy that works well for streams but may fail for other cases. However, we like to place these boxes in the loop because we will show that the feedback is only defined if an initial condition is provided, which can be interpreted as initial values in the registers. Next, there are blue edges with labels $\tau_k$. These edges are parameters of the maps that we cannot do recursion with but have more flexible types. This can be useful if we consider causal maps that have additional inputs and outputs that may not even stem from final coalgebras.

The approach to compositional reasoning for causal maps we propose based on the above ideas is that one starts with a set of known causal maps, obtained either directly as chain maps or the construction we provide in the paper. Then one can build arbitrarily complex compositions and loops around these maps using the formalism of traced monoidal and tensored categories. Once construction and reasoning are done, causal maps can be easily obtained from the chain maps by taking limits. All of this works fairly generally, as long as the assumptions in Section 2.2 are fulfilled and that suitable initial conditions for recursion are provided.

## 1.1 Contributions and Outline

We contribute in Section 4 a framework for working compositionally with chain maps. This framework consists of a construction of string diagrams that differentiate between interfaces for recursion and for parameters. These come about as certain symmetric monoidal, enriched, and tensored categories. For such categories, we show that a trace operator can be obtained relative to the recursion interface of morphisms. To enable the use of this framework, we prove in Section 3 the correspondence between chain maps and causal maps, from which we obtain a very flexible method of composition and recursion for causal maps. We also show in Section 3.1 a third way to define causal maps in terms of a metric that is induced on $\nu F$ by the diagram $\Phi F$. This metric view allows us to understand causality better in certain examples, like streams and partial computations. In Section 5, we discuss applications to probabilistic computations and we pay particular attention to linear maps, which turn out to be automatically causal. Our framework provides then an alternative view on the various calculi for linear circuits. We end with some concluding remarks in Section 6.

Before we begin with the actual work, we recall in the following Section 2 some background on (enriched) monoidal categories and guarded recursion, and we prove some small results to get the theory of the ground.

## 2 Preliminaries

We follow the convention to use boldface letters $\mathbf{C}$ for categories, capital letters such as $X$ for objects, lower case letters for morphisms, capital letters such as $F$ for functors, small Greek letters like $\mu$ for natural transformations, and $\alpha, \beta$ for ordinals. We denote by $\omega$ the ordinal of the natural numbers. Finally, $\sigma, \tau, \gamma$ will be for $\alpha^{\mathrm{op}}$-indexed diagrams in some category.

Recall [11] that a *symmetric monoidal category (SMC)* is a category $\mathbf{C}$ with tensor product functor $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and a tensor unit $I \in \mathbf{C}$ with the associativity, unit and symmetry isomorphisms. An SMC is *closed* if for every object $X \in \mathbf{C}$, the functor $\mathrm{Id} \otimes X\colon \mathbf{C} \to \mathbf{C}$ has a right-adjoint. In particular, a *Cartesian closed category (CCC)* is a closed SMC with products acting as tensor and exponentials as their right adjoint: $- \times X \dashv -^X$. Let $\mathbf{V}$ be a SMC. A $\mathbf{V}$-category $\mathbf{C}$ is a $\mathbf{V}$-enriched category, which means that its morphisms $\mathbf{C}(X, Y)$ are objects in $\mathbf{V}$, and composition and identity are morphisms $c_{X,Y,Z}\colon \mathbf{C}(Y, Z) \otimes \mathbf{C}(X, Y) \to \mathbf{C}(X, Z)$ and $u_X\colon I \to \mathbf{C}(X, X)$ in $\mathbf{V}$ subject to the corresponding associativity and unit axioms [10, 6]. For morphisms $f\colon X \to Y$ in a Cartesian closed category $\mathbf{C}$, we denote by $\ulcorner f \urcorner\colon \mathbf{1} \to Y^X$ the "code" of $f$ given by the Cartesian closure. The CCC $\mathbf{C}$ is a $\mathbf{C}$-category (self-enriched) by taking $\ulcorner \mathrm{id} \urcorner\colon \mathbf{1} \to X^X$ as unit and the composition $\mathrm{comp}_{X,Y,Z}\colon Z^Y \times Y^X \to Z^X$ is given by the exponential adjunction. A functor $F : \mathbf{C} \to \mathbf{C}$ is called *strong* if there is a natural family of morphisms $F_{X,Y}\colon Y^X \to FY^{FX}$, such that $F_{X,Y} \circ \ulcorner f \urcorner = \ulcorner Ff \urcorner$ for all $f\colon X \to Y$. This makes $F$ a $\mathbf{C}$-functor for the self-enrichment of $\mathbf{C}$.

Let $\mathbf{C}$ be a category and $F\colon \mathbf{C} \to \mathbf{C}$ a functor. An *$F$-coalgebra* (or just coalgebra) is a morphism $c\colon X \to FX$ in $\mathbf{C}$. If we need to be explicit about the carrier $X$, we also write $(X, c)$. A coalgebra homomorphism from $(X, c)$ to $(Y, d)$ is a morphism $f\colon X \to Y$ in $\mathbf{C}$, satisfying $Ff \circ c = d \circ f$. A coalgebra $(Y, d)$ is *final* if it is final in the category of $F$-coalgebras $\mathrm{CoAlg}(F)$, i.e., if for every coalgebra $(X, c)$ there exists a unique coalgebra homomorphism from $(X, c)$ to $(Y, d)$.

Given a category $\mathbf{C}$, the *category of descending $\alpha$-chains in* $\mathbf{C}$, here denoted by $\overleftarrow{\mathbf{C}}$, is the functor category $[\alpha^{\mathrm{op}}, \mathbf{C}]$. Objects of $\overleftarrow{\mathbf{C}}$ are functors $\sigma\colon \alpha^{\mathrm{op}} \to \mathbf{C}$, which assign each $i < \alpha$ an object $\sigma_i$ of $\mathbf{C}$ and each pair $i \leq j$ a morphism $\sigma(i \leq j) : \sigma_j \to \sigma_i$ in $\mathbf{C}$. A morphism $f\colon \sigma \to \tau$ in $\overleftarrow{\mathbf{C}}$ is a natural transformation, which means that it is an $\alpha$-indexed family of morphisms such that $f_i \circ \sigma(i \leq j) = \tau(i \leq j) \circ f_j$ holds. Such $f$ will often be called a *chain map* for simplicity. We also record here that the chain category construction gives rise to a 2-functor $\overleftarrow{(-)}\colon \mathbf{Cat} \to \mathbf{Cat}$ on the category of categories. In particular, a functor $F\colon \mathbf{C} \to \mathbf{D}$ gives rise to a functor $\overleftarrow{F}\colon \overleftarrow{\mathbf{C}} \to \overleftarrow{\mathbf{D}}$ by post-composition with diagrams (point-wise application) and similarly for natural transformations. Finally, let us denote by $K\colon \mathbf{C} \to \overleftarrow{\mathbf{C}}$ the constant functor which assigns an object $X$ of $\mathbf{C}$ to the constant chain given by $KX_i = X$ and $KX(i \leq j) = \mathrm{id}_X$. If $\mathbf{C}$ has $\alpha^{\mathrm{op}}$-limits, then we assume them to be given as an adjunction $\langle K \dashv L, \eta, \epsilon \rangle\colon \mathbf{C} \to \overleftarrow{\mathbf{C}}$, where $L\colon \overleftarrow{\mathbf{C}} \to \mathbf{C}$ assigns to a chain its limit.

### 2.1 Domain Theory of Chains

It is well known [1, 8] that if $F\colon \mathbf{C} \to \mathbf{C}$ has a final coalgebra, then there is a limit ordinal $\alpha$ for which $F$ is $\alpha^{\mathrm{op}}$-continuous (preserves limits of $\alpha^{\mathrm{op}}$-diagrams) and the final coalgebra is given by the limit of the so-called final chain. The main tool of this paper is this final chain and we shall therefore recap recursion theory for such chains, see [13, 4, 3].

The category $\overleftarrow{\mathbf{C}}$ of $\alpha^{\mathrm{op}}$-chains has properties that are akin to that of domains used in recursion theory, with the main difference that fixed point theorems require guardedness via the so-called later modality. We assume in what follows that $\mathbf{C}$ is Cartesian closed, which implies that $\overleftarrow{\mathbf{C}}$ is also a CCC, and that $\mathbf{C}$ has sufficiently many limits, cf. Section 2.2.

The *later modality* is a functor $\blacktriangleright\colon \overleftarrow{\mathbf{C}} \to \overleftarrow{\mathbf{C}}$ defined on objects by $(\blacktriangleright \sigma)_i = \lim_{j<i} \sigma_j$ and it comes with a natural transformation $\mathrm{next}\colon \mathrm{Id} \to \blacktriangleright$. Since products preserve limits, there are natural isomorphisms $\delta^{\blacktriangleright}_{\sigma,\tau}\colon \blacktriangleright \sigma \times \blacktriangleright \tau \to \blacktriangleright(\sigma \times \tau)$ and $\varepsilon^{\blacktriangleright}\colon \mathbf{1} \to \blacktriangleright \mathbf{1}$. If $\omega$ is used as indexing ordinal, one can easily show that $(\blacktriangleright \sigma)_0 \cong \mathbf{1}$ and $(\blacktriangleright \sigma)_{n+1} \cong \sigma_n$ via a chain map.

We are interested in the category $\overleftarrow{\mathbf{C}}$ here because it allows us to do so-called guarded recursion, which comes in the form of fixed point solution theorems for morphism and for functors analogue to those occurring in domain theory. However, what differentiates guarded recursion from domain theory is that we only find fixed points of contractive morphisms. A *solution* or fixed point of a morphism $h\colon \tau \times \gamma \to \gamma$ in $\overleftarrow{\mathbf{C}}$ is a morphism $s\colon \tau \to \gamma$ with $s = h \circ \langle \mathrm{id}_\tau, s\rangle$. We call a morphism $h\colon \tau \times \gamma \to \gamma$ *contractive* if there is $g\colon \tau \times \blacktriangleright \gamma \to \gamma$ with $h = g \circ (\mathrm{id}_\tau \times \mathrm{next}_\gamma)$. The main point is now that any contractive morphism $h$ has a solution in $\overleftarrow{\mathbf{C}}$.

The isomorphisms $\delta^{\blacktriangleright}$ and $\varepsilon^{\blacktriangleright}$ make $\blacktriangleright$ a (strong) monoidal functor and thus allow us to change the enriching base and obtain a $\overleftarrow{\mathbf{C}}$-category $\overleftarrow{\mathbf{C}}_{\blacktriangleright}$ with the same objects as $\overleftarrow{\mathbf{C}}$ but $\overleftarrow{\mathbf{C}}_{\blacktriangleright}(\sigma, \tau) = \blacktriangleright(\tau^\sigma)$ as morphism object. The monoidal natural transformation next induces a $\overleftarrow{\mathbf{C}}$-functor $\mathrm{Next}\colon \overleftarrow{\mathbf{C}} \to \overleftarrow{\mathbf{C}}_{\blacktriangleright}$ by putting $N_{\sigma,\tau} = \mathrm{next}_{\tau^\sigma}\colon \tau^\sigma \to \blacktriangleright(\tau^\sigma)$. A $\overleftarrow{\mathbf{C}}$-functor $F\colon \overleftarrow{\mathbf{C}} \to \overleftarrow{\mathbf{C}}$ is called *locally contractive* if there is a $\overleftarrow{\mathbf{C}}$-functor $G\colon \overleftarrow{\mathbf{C}}_{\blacktriangleright} \to \overleftarrow{\mathbf{C}}$ with $G \circ \mathrm{Next} = F$. Explicitly, there is a family of morphisms $G_{\sigma,\tau}\colon \blacktriangleright(\sigma^\tau) \to F\sigma^{F\tau}$ with $F_{\sigma,\tau} = G_{\sigma,\tau} \circ \mathrm{next}_{\sigma^\tau}$, $G_{\sigma,\sigma} \circ \blacktriangleright \ulcorner \mathrm{id} \urcorner \circ \varepsilon^{\blacktriangleright} = \ulcorner \mathrm{id} \urcorner$ and $\mathrm{comp} \circ (C_{\sigma,\tau} \times C_{\gamma,\sigma}) = C_{\gamma,\tau} \circ \blacktriangleright \mathrm{comp} \circ \delta^{\blacktriangleright}$.

Throughout this paper, we will use that $\blacktriangleright$ is locally contractive, and that if $F$ and $G$ are $\overleftarrow{\mathbf{C}}$-functors and at least one of them is locally contractive, then $F \circ G$ is locally contractive. Moreover, we will need the following result.

▶ **Lemma 1.** *Given a functor $F\colon \mathbf{C} \to \mathbf{C}$, the functor $\overleftarrow{F}\colon \overleftarrow{\mathbf{C}} \to \overleftarrow{\mathbf{C}}$ is a $\overleftarrow{\mathbf{C}}$-functor if and only if $F$ is a $\mathbf{C}$-functor.*

What makes locally contractive functor interesting, is that they admit unique fixed points: Given a locally contractive functor $F\colon \overleftarrow{\mathbf{C}} \to \overleftarrow{\mathbf{C}}$, there is a unique chain $\nu F$ with isomorphisms $\mathrm{obs}\colon \nu F \to F(\nu F)$ and $\mathrm{fold} = \mathrm{obs}^{-1}\colon F(\nu F) \to \nu F$. In this paper, we pick coinduction as our main principle and consider $(\nu F, \mathrm{obs})$ as final object in $\mathrm{CoAlg}(F)$.

▶ **Lemma 2.** *There is a functor $\Phi\colon \mathrm{Endo}(\mathbf{C}) \to \overleftarrow{\mathbf{C}}$ given on objects by $\Phi F = \nu\big(\blacktriangleright \overleftarrow{F}\big)$, which exists because $\blacktriangleright \circ \overleftarrow{F}$ is locally contractive. We call $\Phi F$ the* final chain *of $F$.*

**Proof.** Given a natural transformation $\alpha\colon F \to G$, we define $\Phi\alpha$ coinductively as in the following diagram.

$$\begin{array}{ccc}
\Phi F & \xrightarrow{\Phi\alpha} & \Phi G \\
\downarrow{\scriptstyle \mathrm{obs}} & & \\
\blacktriangleright \overleftarrow{F}(\Phi F) & & \downarrow{\scriptstyle \mathrm{obs}} \\
\downarrow{\scriptstyle \blacktriangleright \alpha_{\Phi F}} & & \\
\blacktriangleright \overleftarrow{G}(\Phi F) & \xrightarrow{\blacktriangleright \overleftarrow{G}(\Phi\alpha)} & \blacktriangleright \overleftarrow{G}(\Phi G)
\end{array}$$

Preservation of identities and composition follow by standard arguments from finality. ◀

If $F$ preserves $\alpha^{\mathrm{op}}$-limits, that is, if $L\overleftarrow{F} \cong FL$, then the limit adjunction $K \dashv L$ lifts to an adjunction $\overline{K} \dashv \overline{L}$ with $\overline{K}\colon \mathrm{CoAlg}(F) \to \mathrm{CoAlg}(\blacktriangleright \overleftarrow{F})$, see [3]. In particular, $\overline{L}(\Phi F, \mathrm{obs})$ is a final $F$-coalgebra with carrier $L(\Phi F)$.

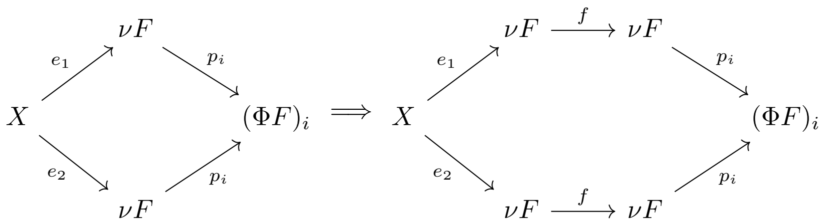
## 2.2 Assumptions

Given the above, we assume the following for the remainder of the paper: $\mathbf{C}$ is a Cartesian closed category; $\alpha$ is a limit ordinal; $\mathbf{C}$ has $\alpha^{\mathrm{op}}$-limits and $\partial(\alpha \downarrow i)^{\mathrm{op}}$-limits, where $\partial(\alpha \downarrow i)$ is the category that contains all $j < i$; $F$ is a strong functor on $\mathbf{C}$ that preserves $\alpha^{\mathrm{op}}$-limits.

# 3 Causality

In this section, we extend the definition of $\omega$-causal operators [14, Def. 8.1] to arbitrary categories but we do not define causal algebra. Although, our definition can be easily extended to causal algebras. For this purpose, we assume that $F$ preserves $\alpha^{\mathrm{op}}$-limits and thus $L\Phi F$ can be taken as the carrier $\nu F$ of a final $F$-coalgebra. We denote by $(\nu F, (p_i)_{i<\alpha})$, the universal cone defining a limit for $\Phi F$ and we define causal morphisms on $\nu F$ as follows.

▶ **Definition 3.** *A morphism $f\colon \nu F \to \nu F$ is causal if for every object $X$ of $\mathbf{C}$, morphisms $e_1, e_2\colon X \to \nu F$ and $i < \alpha$: if $p_i \circ e_1 = p_i \circ e_2$, then $p_i \circ f \circ e_1 = p_i \circ f \circ e_2$. Diagrammatically:*

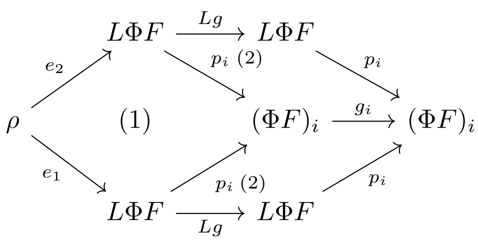



*We denote the set of causal morphisms on $\nu F$ by $\mathrm{Caus}(\nu F, \nu F) \subseteq \mathbf{C}(\nu F, \nu F)$.*

In the following theorem we compare two characterisations of causal morphisms on $\nu F$.

▶ **Theorem 4.** *There is a map $\lambda\colon \overleftarrow{\mathbf{C}}(\Phi F, \Phi F) \to \mathrm{Caus}(\nu F, \nu F)$ with $\lambda(g) = Lg$. If there is a section $s\colon \Phi F \to KL\Phi F$ of $\epsilon_{\Phi F}$ in $\overleftarrow{\mathbf{C}}$, i.e. $\epsilon_{\Phi F} \circ s = \mathrm{id}_{\Phi F}$, then $\lambda$ is an isomorphism.*

**Proof.** We define $\lambda : \overleftarrow{\mathbf{C}}(\Phi F, \Phi F) \to \mathrm{Caus}(\nu F, \nu F)$ such that for each $g\colon \Phi F \to \Phi F$, $\lambda(g) = Lg$. To show that $\lambda(g)$ is causal we need to prove, by Definition 3, that if diagram (1) below commutes, then the outer diagram must also commute, for any $\rho \in \overleftarrow{\mathbf{C}}$ and morphisms $e_1, e_2 : \rho \to \nu F$. In the diagram, we use $L\Phi F$ for $\nu F$.




To prove that the outer diagram commutes, it is enough to prove that diagram (2) commutes. Because of naturality of the counit $\epsilon$ of the adjunction $\langle K \dashv L, \eta, \epsilon \rangle$, the diagram below commutes.

$$\begin{array}{ccc} KL\Phi F & \xrightarrow{KLg} & KL\Phi F \\ {\scriptstyle \epsilon_{\Phi F}}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_{\Phi F}} \\ \Phi F & \xrightarrow{\quad g \quad} & \Phi F \end{array}$$

Hence diagram (2) commutes, as being the $i^{\text{th}}$ component of the above commuting diagram. Therefore, $\lambda(g)$ is causal.

Given the section $s\colon \Phi F \to KL\Phi F$, we define an inverse $\chi : \mathrm{Caus}(\nu F, \nu F) \to \overleftarrow{\mathbf{C}}(\Phi F, \Phi F)$ of $\lambda$ on causal maps $f\colon \nu F \to \nu F$ by letting $\chi(f) = \Phi F \xrightarrow{s} KL\Phi F \xrightarrow{Kf} KL\Phi F \xrightarrow{\epsilon_{\Phi F}} \Phi F$. $\chi(g)$ is a chain map in $\overleftarrow{\mathbf{C}}$ because it is a composition of chain maps in $\overleftarrow{\mathbf{C}}$. We have, $(\chi \circ \lambda)(g) = g$, since the following diagram commutes by naturality of $\epsilon$ and $s$ being a section.

$$\begin{array}{ccccc} \Phi F & \xrightarrow{\quad s \quad} & KL\Phi F & \xrightarrow{KLg} & KL\Phi F \\ & {\scriptstyle \mathrm{id}_{\Phi F}}\searrow & {\scriptstyle \epsilon_{\Phi F}}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_{\Phi F}} \\ & & \Phi F & \xrightarrow{\quad g \quad} & \Phi F \end{array}$$

We also have $(\lambda \circ \chi)(f) = f$: The following diagram commutes because of causality of $f$, naturality of $\eta$, and the triangular axiom of adjunction.

$$\begin{array}{ccccc} LKL\Phi F & \xrightarrow{LKf} & LKL\Phi F & & \\ {\scriptstyle \eta_{L\Phi F}}\big\uparrow & & {\scriptstyle \eta_{L\Phi F}}\big\uparrow & \searrow{\scriptstyle L\epsilon_{\Phi F}} & \\ L\Phi F & \xrightarrow{\quad f \quad} & L\Phi F & \xrightarrow{\mathrm{id}_{L\Phi F}} & L\Phi F \\ {\scriptstyle Ls}\big\downarrow & & & \nearrow{\scriptstyle L\epsilon_{\Phi F}} & \\ LKL\Phi F & \xrightarrow{LKf} & LKL\Phi F & & \end{array}$$

Thus $\lambda$ is an isomorphism with inverse $\chi$. ◀

Importantly, this characterisation allows us to exploit all the domain-theoretic tools that are available in $\overleftarrow{\mathbf{C}}$ to compose and reason about causal morphisms.

Let us pause for a moment to take a look at some examples in the category **Set**. First of all, we note that we generally get the required section in Theorem 4 because the limit projections split if the involved chains are non-empty. Thus, chain and causal maps are equivalent in **Set**. Let us explore more concretely the familiar examples of streams and partial computations.

▶ **Example 5.** Let $S\colon \mathbf{Set} \to \mathbf{Set}$ be the functor defined by $S(X) = R \times X$, for some set $R$. The set $R^\omega$ consists of streams over $R$, defined by $R^\omega = [\mathbb{N}, R]$. If we use $\omega$ as ordinal for indexing, then the final chain $\Phi S$ is isomorphic to the following chain.

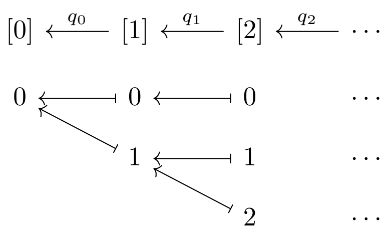
$$\mathbf{1} \xleftarrow{\;!\;} R \xleftarrow{\pi_1} R^2 \xleftarrow{\pi_2} R^3 \longleftarrow \cdots$$

That is, $(\Phi S)_0 \cong \mathbf{1}$ and for every $i \in \mathbb{N}$, $(\Phi S)_i \cong R^i$ via a chain map. Indeed, $L\Phi S \cong R^\omega$ with the projections $(p_i)_{i\in\mathbb{N}}$, such that $p_i\colon R^\omega \to R^i$ giving for every $s \in R^\omega$ its first $i$ elements. It is well known [7] that a function $f : R^\omega \to R^\omega$ is causal if and only if for all $k \in \mathbb{N}$, $s, t \in R^\omega$, if $s(i) = t(i)$ for all $i \le k$, then $f(s)(k) = f(t)(k)$. Which implicitly includes every $i \le k$, that is $f(s)(i) = f(t)(i)$, and that is exactly Definition 3. From Theorem 4, we now obtain that we can equivalently see $f$ as a chain map $\chi(f)\colon \Phi S \to \Phi S$, where for $u \in R^n$ we have $\chi(f)_{n+1}(u) = f(u : s)$ for any stream $s \in R^\omega$. Note that this requires that $R$ is inhabited.

▶ **Example 6.** For the functor $N \colon \mathbf{Set} \to \mathbf{Set}$ given by $N(X) = X + \mathbf{1}$, where $\mathbf{1} = \{*\}$, one has $\nu N \cong \mathbb{N} \cup \{\infty\}$. and we use $\omega$ as indexing ordinal. The final chain $\Phi N$ is isomorphic to the following chain, in which $[n] = \{k \in \mathbb{N} \mid 0 \leq k < n\}$.

$$[0] \xleftarrow{\;!\;} [1] \xleftarrow{\;q_1\;} [2] \xleftarrow{\;q_2\;} [3] \longleftarrow \cdots$$

The projections $q_i$ are the identity on numbers below $i$ and truncate all higher numbers. Pictorially this looks as follows.



One can show [14, Ex. 8.4] that a map $f \colon \nu N \to \nu N$ is causal if for all $n, m$ and $i \leq \min(n, m)$, then $f(n) = f(m)$ or $i \leq \min(f(n), f(m))$.

One may wonder where the last condition in Example 6 comes from. Let us, therefore, digress for a moment and explore yet another characterisation of causal morphisms.

## 3.1 Causality and Metric Maps

For the purpose of comparing causal maps with metric maps, we assume additionally that $\mathbf{C}$ is locally small and that it has a *generator* $G$, which is an object such that the hom-functor $\mathbf{C}(G, -) \colon \mathbf{C} \to \mathbf{Set}$ is faithful. We will denote this functor by $E = \mathbf{C}(G, -)$ and its action on a morphism $f \colon X \to Y$ by $f_* \colon EX \to EY$. One can think of $x \in EX$ as element of $X$ and $f_*(x) \in EX$ as its image under $f$. Moreover, we need that the functor $F$ is $\omega^{\mathrm{op}}$-continuous. These assumptions allow us to define a metric on final coalgebras and then prove that metric maps correspond to causal maps.

Let $d \colon E(\nu F) \times E(\nu F) \to [0, 1]$ be the metric defined for $e_1, e_2 \in E(\nu F)$ as follows.

$$d(e_1, e_2) = \sup\left\{2^{-i} \;\middle|\; p_i \circ e_1 \neq p_i \circ e_2, i \in \mathbb{N}\right\} = \inf\left\{2^{-i} \;\middle|\; p_i \circ e_1 = p_i \circ e_2, i \in \mathbb{N}\right\}$$

One can easily observe from Definition 3 that two outputs of causal morphisms $f_*$ should not be more distant than their corresponding inputs. That is, causal functions are metric maps, in the following sense.

▶ **Definition 7.** *Let $(X, d_X), (Y, d_Y)$ be two metric spaces. A function $f : X \to Y$ is a* metric map *when for any elements $x, y \in X$, the following condition is fulfilled.*

$$d_Y(f(x), f(y)) \leq d_X(x, y)$$

*Metric spaces and metric maps form a category* **Met** .

Now we can show the correspondence between causal morphisms and metric maps.

▶ **Theorem 8.** *The following are equivalent:*

1. $f \in \mathrm{Caus}(\nu F, \nu F)$
2. $f \in \mathbf{Met}((\nu F, d), (\nu F, d))$

**Proof.** ($\mathbf{1} \to \mathbf{2}$) By the universal property of sup, we need to prove $2^{-l} \leq d(x, y)$ for all $l$ with $p_l \circ f_*(x) \neq p_l \circ f_*(y)$. Given such an $l$, we get by causality of $f$ that $p_l \circ x \neq p_l \circ y$ and hence $2^{-l} \leq d(x, y)$. As this holds for all $l$, we get $d(f_*(x), f_*(y)) \leq d(x, y)$.

($\mathbf{2} \to \mathbf{1}$) Conversely, let us assume that $f$ is a metric map. That is $d(f_*(x), f_*(y) \leq d(x, y)$, which implies that $l \geq k$. Hence, we have for all $i < k$ the following.

$$p_i \circ x = p_i \circ y \Longrightarrow f \circ p_i \circ x = f \circ p_i \circ y$$

Since $f$ is a metric map, we also have $p_i \circ f_*(x) = p_i \circ f_*(y)$. Thus $f$ is causal. ◀

Birkedal et al. [4] show that there is an adjunction between certain metric spaces and $\overleftarrow{\mathbf{Set}}$, and that there is a one-to-one correspondence between contractive maps in the metric sense and contractive maps in $\overleftarrow{\mathbf{Set}}$, see Section 2.1. One can think of Theorem 8 as a partial generalisation of this result, although we are mostly interested in it here to understand causality better in some examples.

▶ **Example 9.** Recall that we cited in Example 6 a rather odd looking characterisation of causal maps on partial computations. We can derive this characterisation from Theorem 8 as follows. Since if $n = m$ we must have $f(n) = f(m)$, suppose without loss of generality $n \neq m$. For $i \leq \min(n, m)$, we get $d(n, m) = 2^{-(\min(n,m)+1)}$. If $f$ is causal, we either have $f(n) = f(m)$ or $d(f(n), f(m)) = 2^{-(\min(f(n),f(m))+1)} \leq d(n, m)$. By inspecting the two sides, we get that $i \leq \min(n, m) \leq \min(f(n), f(m))$, which is what we wanted to prove.

The results in Theorem 4 and Theorem 8 can be summed up as in the following diagram.

$$\begin{array}{ccc} & \mathrm{Caus}(\nu F, \nu F) & \\ \overset{\cong}{\swarrow\!\!\nearrow} & & \overset{\cong}{\searrow\!\!\nwarrow} \\ \overleftarrow{\mathbf{C}}(\Phi F, \Phi F) & \underset{\cong}{\longleftrightarrow} & \mathrm{Met}((\nu F, d), (\nu F, d)) \end{array}$$

## 4 Composition and Recursion

In this section, we construct for a fixed chain $\sigma$ a symmetric monoidal category $\mathbf{P}_\sigma$ together with a trace-like operator. This category allows us to construct diagrams of arbitrary causal morphisms with feedback loops. The SMC $\mathbf{P}_\sigma$ will have as morphisms something one may think of building blocks with two kinds of interfaces: one for things of type $\sigma$ over which we do recursion via traces and one type for parameter of arbitrary type. The diagram in Figure 1 displays the kind of circuit that we intend to build. Here, we build a circuit out of two causal morphisms $f$ and $g$, where $\tau_k$ are types of the parameters (blue wires) and the three loops going through small boxes indicate recursive feedback that goes through a *register* that can store elements of type $\sigma$ (black wires). Such diagrams can be built, in the usual way, by parallel and sequential composition of morphisms and by looping interfaces of type $\sigma$ back to inputs. What is not allowed are loops of types other than $\sigma$.

Let us first explain the nature of $\mathbf{P}_\sigma$ and then we prove that it is a traced SMC. Recall that we can associate to any SMC, in this case, $\overleftarrow{\mathbf{C}}$, a canonical PROP [12] $\mathbf{H}_\sigma$ with objects being natural numbers and morphisms given by $\mathbf{H}_\sigma(n, m) = \overleftarrow{\mathbf{C}}(\sigma^n, \sigma^m)$. In fact, any PROP is of this form [2]. In $\mathbf{H}_\sigma$, we could build diagrams with only black wires and our result Corollary 17 below will have as special case that this category is a traced SMC. However, we wish to have the extra flexibility of additional parameters, which we can achieve by creating a symmetric monoidal $\overleftarrow{\mathbf{C}}$-category that is tensored over $\overleftarrow{\mathbf{C}}$.

▶ **Theorem 10.** *Let* $(\mathbf{V}, \otimes, I)$ *be a closed SMC and* $v \in \mathbf{V}$ *some object. Denote by* $\mathbf{H}_v$ *the* $\mathbf{V}$*-enriched PROP with natural numbers as objects and morphisms* $v^{\otimes n} \to v^{\otimes m}$ *where* $v^{\otimes n}$ *is the* $n$*-fold tensor product of* $v$*. There is a* $\mathbf{V}$*-enriched SMC* $\mathbf{P}_v$ *with a fully faithful monoidal* $\mathbf{V}$*-functor* $\underline{(-)}\colon \mathbf{H}_v \to \mathbf{P}_v$ *that is* tensored *over* $\mathbf{V}$*, which means that there is a monoidal functor* $\odot\colon \mathbf{V} \times \mathbf{P}_v \to \mathbf{P}_v$ *with natural isomorphisms* $\mathbf{P}_v(u \odot X, Y) \cong \mathbf{V}(u, \mathbf{P}_v(X, Y))$ *for* $u \in \mathbf{V}$ *and* $X, Y \in \mathbf{P}_v$*.*

**Proof.** We define $\mathbf{P}_v$ to have as objects pairs $(u, n)$ with $u \in \mathbf{V}$ and $n \in \mathbb{N}$, and as morphisms we take

$$\mathbf{P}_v((u, n), (w, m)) = \mathbf{V}\big(u \otimes v^{\otimes n}, w \otimes v^{\otimes m}\big)\,.$$

Since $\mathbf{V}$ is closed, this makes $\mathbf{P}_v$ immediately a $\mathbf{V}$-category. It is also symmetric monoidal with the product $(u, n) \otimes_{\mathbf{P}_v} (w, m) = (u \otimes w, n + m)$ and unit $I_{\mathbf{P}_v} = (I, 0)$. The functor $\mathbf{H}_v \to \mathbf{P}_v$ is given by $\underline{n} = (I, n)$ and $\underline{f} = I \otimes f$. It is obviously monoidal and faithful, and that it is full follows from $I$ being the monoidal unit. Finally, the tensor is defined by $u \odot (w, n) = (u \otimes w, n)$ and we get immediately

$$\begin{aligned}
\mathbf{P}_v(u \odot (x, n), (y, m)) &= \mathbf{P}_v((u \otimes x, n), (y, m)) \\
&= \mathbf{V}\big(u \otimes x \otimes v^{\otimes n}, y \otimes v^{\otimes m}\big) \\
&\cong \mathbf{V}\big(u \otimes, \mathbf{V}\big(x \otimes v^{\otimes n}, y \otimes v^{\otimes m}\big)\big) \\
&= \mathbf{V}(u \otimes, \mathbf{P}_v((x, n), (y, m)))
\end{aligned}$$

by $\mathbf{V}$ being closed. Thus $\mathbf{P}_v$ is also tensored over $\mathbf{V}$. ◀

We now apply Theorem 10 to our situation of $\alpha^{\mathrm{op}}$-chains to obtain for $\sigma \in \overleftarrow{\mathbf{C}}$ a $\overleftarrow{\mathbf{C}}$-category $\mathbf{P}_\sigma$ with pairs $(\tau, n)$ of $\tau \in \overleftarrow{\mathbf{C}}$ and $n \in \mathbb{N}$ and

$$\mathbf{P}_\sigma((\tau, n), (\gamma, m)) = \overleftarrow{\mathbf{C}}(\tau \times \sigma^n, \gamma \times \sigma^m)$$

as hom-objects. We denote the monoidal product of $\mathbf{P}_\sigma$ simply by $\otimes$ and its unit by I. Since morphisms in $\mathbf{P}_\sigma$ are particular morphisms in $\overleftarrow{\mathbf{C}}$, we make no distinction between, e.g., $\mathrm{id}_{(\tau,n)}$ and $\mathrm{id}_{\tau \times \sigma^n}$ to lighten notation a bit.

Our goal now is to enable recursion in $\mathbf{P}_\sigma$ via a trace operator [9]. Except that our trace will be *relative to* $\mathbf{H}_\sigma$ in the sense that there is a family of maps

$$\mathrm{Tr}^k_{X,Y}\colon \mathbf{P}_\sigma(X \otimes \underline{k}, Y \otimes \underline{k}) \to \mathbf{P}_\sigma(X, Y)$$

indexed by $X, Y \in \mathbf{P}_\sigma$ and $k \in \mathbf{H}_\sigma$ that fulfils the usual trace axioms. Since the functor $\mathbf{H}_\sigma \to \mathbf{P}_\sigma$ is fully faithful, this will expose $\mathbf{H}_\sigma$ as a proper traced SMC.

Whenever morphisms are defined by recursive equations, one has to provide boundary conditions to obtain a well-defined solution to the equations, even if they are implicit. In analogy with registers to create well-defined feedback loops as in Figure 1, an initial value that we place in the registers will take the role of boundary conditions in our case.

▶ **Definition 11.** *We call a morphism* $\mathrm{i}\colon \blacktriangleright\sigma \to \sigma$ *in* $\overleftarrow{\mathbf{C}}$ *an* initial value*. It gives rise to a morphism on powers of* $\sigma$ *by* $\hat{\mathrm{i}}^k = \blacktriangleright\big(\sigma^k\big) \xrightarrow{\delta^{\blacktriangleright}} (\blacktriangleright\sigma)^k \xrightarrow{\mathrm{i}^k} \sigma^k$*. A morphism* $g\colon n \to m$ *in* $\mathbf{H}_\sigma$ *is* compatible *with* i *if* $\hat{\mathrm{i}}^m \circ \blacktriangleright g = g \circ \hat{\mathrm{i}}^n$*.*

If $\sigma \in [\omega^{\mathrm{op}}, \mathbf{C}]$, then an initial value $i\colon \blacktriangleright\sigma \to \sigma$ consists of morphisms $i_0\colon \mathbf{1} \to \sigma_0$ and $i_{n+1}\colon \sigma_n \to \sigma_{n+1}$ that are compatible with the chain $\sigma$. In the case of streams, see Example 5,

$i\colon \blacktriangleright(\Phi S) \to \Phi S$ picks out an element $i_1\colon \mathbf{1} \to R$ that all $i_k\colon R^k \to R^{k+1}$ have to return as the first element. Compatibility of $g$ with $i$ means then that $g_1 \circ i_1 = i_1$, which is for example the case when $i_1$ returns 0 and $g$ is linear, see Section 5.1.

A good source of initial values for the final chain is pointed functors.

▶ **Proposition 12.** *If $F\colon \mathbf{C} \to \mathbf{C}$ is a pointed functor, i.e., comes with a natural transformation $\eta\colon \mathrm{Id} \to F$, then there is an initial value $\blacktriangleright \Phi F \to \Phi F$.*

**Proof.** The initial value is defined as the composite $\blacktriangleright \Phi F \xrightarrow{\blacktriangleright \overleftarrow{\eta}_{\Phi F}} \blacktriangleright \overleftarrow{F}\Phi F \xrightarrow{\text{fold}} \Phi F$. ◀

In what follows, we assume an initial value to be given and construct the trace relative to it. Since $\overleftarrow{\mathbf{C}}$ is Cartesian closed, we find that the morphism involved in our relative trace has a special shape.

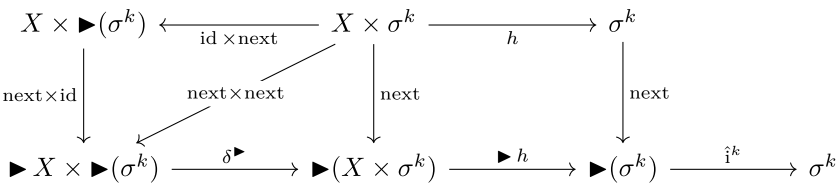
We give the definition of morphisms with $k$-feedback loops as follows.

▶ **Definition 13.** *A $k$-feedback morphism $f \in \mathbf{P}_\sigma((\tau, n) \otimes_{\mathbf{P}_\sigma} \underline{k}, (\gamma, m) \otimes_{\mathbf{P}_\sigma} \underline{k})$ is of the form*

$$f = \langle f_{\mathrm{out}}, f_{\mathrm{fb}} \rangle$$

*such that $f_{\mathrm{out}} \in \mathbf{P}_\sigma((\tau, n) \otimes_{\mathbf{P}_\sigma} \underline{k}, (\gamma, m))$ refers to the output of $f$ and $f_{\mathrm{fb}} \in \mathbf{P}_\sigma((\tau, n) \otimes_{\mathbf{P}_\sigma} \underline{k}, \underline{k})$ refers to the $k$-feedback loops of $f$, given by $f_{\mathrm{fb}} = \hat{\imath}^k \circ \mathrm{next}_{\sigma^k} \circ f_{\mathrm{fb}}$, where $\hat{\imath}^k \in \mathbf{P}_\sigma(\underline{k}, \underline{k})$ such that $(\hat{\imath}^k)_i\colon (\sigma_i)^k \to (\sigma_{i+1})^k$.*

The first step to defining a trace operator is to figure out the behaviour of the register involved in a feedback loop. To this end, let $h\colon (\tau, n) \otimes \underline{k} \to \underline{k}$ be a morphism in $\mathbf{P}_\sigma$ and consider the morphism $\hat{\imath}^k \circ \mathrm{next}_{\sigma^k} \circ h\colon \tau \times \sigma^n \times \sigma^k \to \sigma^k$, which is contractive with $\hat{\imath}^k \circ \blacktriangleright h \circ \delta^{\blacktriangleright} \circ (\mathrm{next}_{\tau\times\sigma^n} \times \mathrm{id})$ because next is a monoidal natural transformation, as the following diagram shows, where $X = \tau \times \sigma^n$.




We denote by $s(h) : (\tau, n) \to \underline{k}$ a solution for $\hat{\imath}^k \circ \mathrm{next}_{\sigma^k} \circ h$, that is, the unique morphism fulfilling the following equation.

$$s(h) = \hat{\imath}^k \circ \mathrm{next}_{\sigma^k} \circ h \circ \langle \mathrm{id}_{(\tau,n)}, s(h) \rangle \tag{1}$$

We collect some properties of $s(h)$ that we need to prove the trace axioms.

▶ **Lemma 14.** *For any $h : (\tau, n) \otimes \underline{k} \to \underline{k}$ and $g : (\tau', n') \to (\tau, n)$ morphisms in $\mathbf{P}_\sigma$, if $s(h)$ is a solution for $\hat{\imath}^k \circ \mathrm{next}_{\sigma^k} \circ h$, then $s(h) \circ g$ is a solution for $\hat{\imath}^k \circ \mathrm{next}_{\sigma^k} \circ h \circ (g \times \mathrm{id}_{\underline{k}})$.*

**Proof.** $s(h) \circ g$ is a solution for $\hat{\imath}^k \circ \mathrm{next}_{\sigma^k} \circ h \circ (g \times \mathrm{id}_{\underline{k}})$, because

$$\begin{aligned}
s(h) \circ g &= \hat{\imath}^k \circ \mathrm{next}_{\sigma^k} \circ h \circ \langle \mathrm{id}_{(\tau,n)}, s(h) \rangle \circ g && \text{by def. of } s(h) \\
&= \hat{\imath}^k \circ \mathrm{next}_{\sigma^k} \circ h \circ \langle \mathrm{id}_{(\tau,n)} \circ g, s(h) \circ g \rangle \\
&= \hat{\imath}^k \circ \mathrm{next}_{\sigma^k} \circ h \circ \langle g \circ \mathrm{id}_{(\tau',n')}, s(h) \circ g \rangle \\
&= \hat{\imath}^k \circ \mathrm{next}_{\sigma^k} \circ h \circ (g \times \mathrm{id}_{\underline{k}}) \circ \langle \mathrm{id}_{(\tau',n')}, s(h) \circ g \rangle
\end{aligned}$$ ◀

The following lemma will allow us to prove the sliding axiom for tracing, but only for chain maps that are compatible with the initial value.

▶ **Lemma 15.** *Suppose* $h' \colon (\tau, n) \otimes \underline{k} \to \underline{k'}$ *and* $g \colon \underline{k'} \to \underline{k}$ *that is compatible with* $\mathrm{i}$*. If* $s(h' \circ (\mathrm{id}_{(\tau,n)}) \otimes g)$ *is a solution for* $\hat{\mathrm{i}}^{k'} \circ \mathrm{next}_{\sigma^{k'}} \circ h' \circ (\mathrm{id}_{(\tau,n)} \otimes g)$*, then* $g \circ s(h' \circ (\mathrm{id}_{(\tau,n)}) \otimes g)$ *is a solution for* $\hat{\mathrm{i}}^{k} \circ \mathrm{next}_{\sigma^{k}} \circ g \circ h'$*.*

**Proof.** Let $s^{k'} = s(h' \circ (\mathrm{id}_{(\tau,n)}) \otimes g)$, then $g \circ s^{k'}$ is a solution for $\hat{\mathrm{i}}^{k} \circ \mathrm{next}_{\sigma^{k}} \circ g \circ h'$, because

$$\begin{aligned} g \circ s^{k'} &= g \circ \hat{\mathrm{i}}^{k'} \circ \mathrm{next}_{\sigma^{k'}} \circ h' \circ \langle \mathrm{id}_{(\tau,n)}, g \circ s^{k'} \rangle, \\ &= \hat{\mathrm{i}}^{k} \circ \blacktriangleright g \circ \mathrm{next}_{\sigma^{k'}} \circ h' \circ \langle \mathrm{id}_{(\tau,n)}, g \circ s^{k'} \rangle && g \text{ compatible with } \mathrm{i} \\ &= \hat{\mathrm{i}}^{k} \circ \mathrm{next}_{\sigma^{k}} \circ g \circ h' \circ \langle \mathrm{id}_{(\tau,n)}, g \circ s^{k'} \rangle \end{aligned}$$

◀

We propose a definition of a trace in $\mathbf{P}_\sigma$ in the following theorem, followed by a proof that it satisfies the axioms of a trace [9].

▶ **Theorem 16.** *For any* $X, Y, \underline{k} \in \mathbf{P}_\sigma$*, we define* $\mathrm{Tr}^{k}_{X,Y} \colon \mathbf{P}_\sigma(X \otimes \underline{k}, Y \otimes \underline{k}) \to \mathbf{P}_\sigma(X, Y)$ *by*

$$\mathrm{Tr}^{k}_{X,Y}(f) = f_{\mathrm{out}} \circ \langle \mathrm{id}_X, s(f_{\mathrm{fb}}) \rangle \tag{2}$$

*a family of morphisms that satisfy the axioms of a trace, with the exception that dinaturality is relative to* $\mathrm{i}$*-compatible morphisms.*

**Proof.** It is required for $\mathrm{Tr}^{k}_{(\tau,n),(\gamma,m)}$ to be a natural transformation on each variable $(\tau, n), (\gamma, m)$ and a dinatural trasformation on $\underline{k}$. The proof of the axioms of trace are done using uniqueness of fixed points.

1. *Naturality on* $(\tau, n)$*:* $\mathrm{Tr}^{k}_{-,(\gamma,m)} \colon \mathbf{P}_\sigma(- \otimes \underline{k}, (\gamma, m) \otimes \underline{k}) \to \mathbf{P}_\sigma(-, (\gamma, m))$ is a natural transformation.
   Let $f \colon (\tau, n) \otimes \underline{k} \to (\gamma, m) \otimes \underline{k}$ be $k$-feedback and $g \colon (\tau', n') \to (\tau, n)$, both morphisms in $\mathbf{P}_\sigma$. We need to show that

$$\mathrm{Tr}^{k}_{(\tau',n'),(\gamma,m)}(f \circ (g \otimes \mathrm{id}_{\underline{k}})) = \mathrm{Tr}^{k}_{(\tau,n),(\gamma,m)}(f) \circ g, \tag{3}$$

   By proving the equality $s(f_{\mathrm{fb}} \circ (g \otimes \mathrm{id}_{\underline{k}})) = s(f_{\mathrm{fb}}) \circ g$.
2. *Naturality on* $(\gamma, m)$*:* $\mathrm{Tr}^{k}_{(\tau,n),-} \colon \mathbf{P}_\sigma((\tau, n) \otimes \underline{k}, - \otimes \underline{k}) \to \mathbf{P}_\sigma((\tau, n), -)$ is a natural transformation.
   Let $f \colon (\tau, n) \otimes \underline{k} \to (\gamma, m) \otimes \underline{k}$ and $g \colon (\gamma, m) \to (\gamma', m')$, we need to show that

$$\mathrm{Tr}^{k}_{(\tau,n),(\gamma',m')}((g \otimes \mathrm{id}_{\underline{k}}) \circ f) = g \circ \mathrm{Tr}^{k}_{(\tau,n),(\gamma,m)}(f) \tag{4}$$

   just by unfolding the definition in $g \circ \mathrm{Tr}^{k}_{(\tau,n),(\gamma,m)}(f)$ .
3. *Dinaturality on* $\underline{k}$*:* $\mathrm{Tr}^{-}_{(\tau,n),(\gamma,m)} \colon \mathbf{P}_\sigma((\tau, n) \otimes -, (\gamma, m) \otimes -) \to \mathbf{P}_\sigma((\tau, n), (\gamma, m))$ is a dinatural transformation, on the full subcategory $\mathbf{H}_\sigma$ with objects of the form $\underline{n} = (K\mathbf{1}, n)$ for all $n \in \mathbb{N}$, and if $\mathrm{i}_{\sigma^{\underline{k}}}$ at every $k \in \mathbb{N}$ satisfies for each $g \colon \underline{k} \to \underline{k}'$, $g \circ \hat{\mathrm{i}}^{k} = \hat{\mathrm{i}}^{k'} \circ \blacktriangleright g$.
   Let $f \colon (\tau, n) \otimes \underline{k} \to (\gamma, m) \otimes \underline{k'}$ and $g \colon \underline{k'} \to \underline{k}$, we need to show that

$$\mathrm{Tr}^{k}_{(\tau,n),(\gamma,m)}((\mathrm{id}_{(\gamma,m)} \otimes g) \circ f) = \mathrm{Tr}^{k'}_{(\tau,n),(\gamma,m)}(f \circ (\mathrm{id}_{(\tau,n)} \otimes g)). \tag{5}$$

   This is done by showing that $s(f_{k'} \circ (\mathrm{id}_{(\tau,n)} \otimes g)) = s(g \circ f_{k'})$.
4. *Vanishing 1:* Let $f : (\tau, n) \otimes \underline{0} \to (\gamma, m) \otimes \underline{0}$ and $\iota_r : - \otimes \underline{1} \to -$, where $\iota_r$ is the right unitor. Then we need to show, that

$$\mathrm{Tr}^{0}_{(\tau,n),(\gamma,m)}(f) = \iota_{r(\gamma,m)} \circ f \circ \iota^{-1}_{r(\tau,n)}. \tag{6}$$

   Note that $f$ is a 0-feedback morphism, therefore $f_{\mathrm{out}} = f$. Hence the equality.

5. *Vanishing 2:* Let $f\colon (\tau,n)\otimes \underline{1}\otimes \underline{1} \to (\gamma,m)\otimes \underline{1}\otimes \underline{1}$. Then

$$\mathrm{Tr}^2_{(\tau,n),(\gamma,m)}(f) = \mathrm{Tr}^1_{(\tau,n),(\gamma,m)}(\mathrm{Tr}^1_{(\tau,n+1),(\gamma,m+1)}(f)) \tag{7}$$

6. *Superposing:* Let $f\colon (\tau,n)\otimes \underline{1} \to (\gamma,m)\otimes \underline{1}$ and $g\colon (\tau',n') \to (\gamma',m')$. The following holds.

$$g \circ \mathrm{Tr}^1_{(\tau,n),(\gamma,m)}(f) = \mathrm{Tr}^1_{(\tau',n')\otimes(\tau,n),(\gamma',m')\otimes(\gamma,m)}(g\otimes f). \tag{8}$$

7. *Yanking:* For the component at $(\underline{1},\underline{1})$ of the braiding, i.e. $\xi_{\underline{1},\underline{1}}$,

$$\mathrm{Tr}^1_{(\underline{1},\underline{1})}(\xi_{\underline{1},\underline{1}}) = \mathrm{id}_{\underline{1}} \tag{9}$$

which holds because $\xi_{\underline{1},\underline{1}} = \langle \pi_1, \pi_2\rangle$ and $id_{\underline{1}}$ is a solution for $\pi_2$

◀

The following is a consequence of Theorem 16.

▶ **Corollary 17.** $\mathrm{Tr}^k_{\underline{n},\underline{m}}$ *is a trace operator on* $\mathbf{H}_\sigma$ *if all* $g\colon k \to k$ *are* i*-compatible.*

**Proof.** This follows from Theorem 16 because the functor $\mathbf{H}_\sigma \to \mathbf{P}_\sigma$ is fully faithful. ◀

Going back to causality, by definition $\mathrm{Tr}^k_{(\tau,n),(\gamma,m)}(f)$ is a morphism in $\overleftarrow{\mathbf{C}}$. Therefore $L(\mathrm{Tr}^k_{(\tau,n),(\gamma,m)}(f))$ is causal by Theorem 4. As Theorem 4 establishes a bijective correspondence, we find that $\mathrm{Caus}(\nu F, \nu F)$ is closed under sequential composition, parallel composition and under recursion via trace. In the following section, we show some applications of this.

# 5 Applications

Before we come to concrete applications, we mention here that *distributive laws*, that is, natural transformations $\delta\colon GF \to FG$, induce morphisms $\hat{\delta}\colon \overleftarrow{G}\Phi F \to \Phi F$ [3]. In particular, distributive laws $\delta\colon \Sigma_n F \to F\Sigma_n$ for the functor $\Sigma_n\colon \mathbf{C}\to\mathbf{C}$ given by $\Sigma_n(X) = X^n$ allow us to define $n$-ary causal morphisms. If, moreover, $F$ is pointed with $\eta\colon \mathrm{Id} \to F$ and $\delta \circ \Sigma_n \eta = \eta \Sigma_n$, the induced map $\hat{\delta}\colon (\Phi F)^n \to \Phi F$ is compatible with the initial value induced by $\eta$, see Proposition 12.

## 5.1 Linear Stream Functions

In this section, we look into functions over the set $R^\omega$ of all streams over a commutative ring $(R, +, ., 0, 1)$. The set $R^\omega$ is a commutative ring, with the pointwise addition $+$, the convolution product $\times$, together with their respective unit stream, see [15]. Moreover, for any $n \in \mathbb{N}$, $(R^\omega)^n$ is an $R^\omega$-module and module homomorphisms are $R^\omega$-linear systems in the following sense.

▶ **Definition 18.** *A system* $\langle f_1, \cdots, f_m\rangle\colon (R^\omega)^n \to (R^\omega)^m$ *is* $R^\omega$*-linear if for every* $i \in \{1,\cdots,m\}$, $f_i\colon (R^\omega)^n \to R^\omega$ *is* $R^\omega$*-linear, i.e., for all streams* $u, v \in R^\omega$ *and* $(s_1,\cdots,s_n),(t_1,\cdots,t_n) \in (R^\omega)^n$

$$f((u\times(s_1,\cdots,s_n)) + (v\times(t_1,\cdots,t_n))) = (u\times f(s_1,\cdots,s_n)) + (v\times f(t_1,\cdots,t_n))$$

*where* $f(s_1,\cdots,s_n) = (z_1\times s_1)+\cdots+(z_n\times s_m)$ *for some fixed rational streams*[1] $z_1,\cdots,z_n \in R^\omega$.

[1] A rational stream is a product of polynomial streams and inverse of a polynomial stream, see [15, Def. 3.32].

We consider the above linear systems because they are characterization of finite stream circuits, possibly with feedback loops under the condition that each loop passes through at least one register, see [15].

▶ **Theorem 19.** *Every linear stream operator* $f : (R^\omega)^n \to R^\omega$ *is causal.*

**Proof.** For every $(s_1, \cdots, s_n), (t_1, \cdots, t_n) \in (R^\omega)^n$, $z_1, \cdots, z_n \in R^\omega$ and $k \in \mathbb{N}$, we assume for all $i \leq k$ that $s_1(i) = t_1(i), \cdots, s_n(i) = t_n(i)$. To prove that $f$ is causal, we need to show that $f(s_1, \cdots, s_n)(k) = f(t_1, \cdots, t_n)(k)$ .

We have the following.

$$f(s_1, \cdots, s_n)(k) = \sum_{j=1}^{n} \sum_{i=0}^{k} z_j(i) \cdot s_j(k-i) \qquad \text{and}$$

$$f(t_1, \cdots, t_n)(k) = \sum_{j=1}^{n} \sum_{i=0}^{k} z_j(i) \cdot t_j(k-i)$$

For all $i \in \{0, \cdots, k\}$, $k - i \leq k$. Hence, $s_j(k-i) = t_j(k-i)$ for all $j \in \{1, \cdots, n\}$. Thus, for all $k \in \mathbb{N}$, the following.

$$f(s_1, \cdots, s_n)(k) = f(t_1, \cdots, t_n)(k) \tag{10}$$

◀

We have seen that $R^\omega \cong L\Phi S$ where $\Phi S$ is isomorphic to an $\omega^{\mathrm{op}}$-chain as described in Example 5. We aim to define stream circuits with feedback loops with initial condition [15] as the trace of functions on the final chain $\Phi S$.

Consider the pointed functor $(S, \eta^S)$, where $S = R \times \mathrm{Id}$, the functor from Example 5 and $\eta^S \colon \mathrm{Id} \to S$ is a natural transformation defined for a fixed $r \in R$ such that $\mu_X(u) = (r, u)$, for every $u \in X$ . Then we get a chain map $\mathrm{i} \colon \blacktriangleright \Phi S \to \Phi S$ defined by $\mathrm{i}_0 \colon \mathbf{1} \to R$ and $\mathrm{i}_n \colon R^n \to R^{n+1}$ with $\mathrm{i}_n(u) = (r, u)$ for every $n \in \mathbb{N}$ and $u \in R^n$. Moreover, $(\pi_n \circ \mathrm{i}_n)(u) = (r, \pi_{n-1}(u))$ as given in the following.

$$\begin{array}{ccccccccc}
\mathbf{1} & \xleftarrow{!} & \mathbf{1} & \xleftarrow{!} & R & \xleftarrow{\pi_1} & R^2 & \longleftarrow & \cdots \\
\downarrow{\scriptstyle !} & & \downarrow{\scriptstyle \mathrm{i}_0} & & \downarrow{\scriptstyle \mathrm{i}_1} & & \downarrow{\scriptstyle \mathrm{i}_2} & & \\
\mathbf{1} & \xleftarrow[!]{} & R & \xleftarrow[\pi_1]{} & R^2 & \xleftarrow[\pi_2]{} & R^3 & \longleftarrow & \cdots
\end{array}$$

The morphism $\mathrm{next} \colon \Phi S \to \blacktriangleright \Phi S$ is defined for every $n \in \mathbb{N}$ by $\mathrm{next}_n : R^{n+1} \to R^n$ such that $\mathrm{next}_n = \pi_n$. Hence, for every $u \in R^{n+1}$, $(\mathrm{i}_n \circ \mathrm{next}_n)(u) = (r, \pi_{n-1}(u))$. Note that, for $r = 0$ the latter can be implemented by a register with initial value 0 [15] and the trace of a function $f \colon (\Phi S)^{n+1} \to (\Phi)^{m+1}$, given by $f = \langle f_{\mathrm{out}}, f_{\mathrm{fb}} \rangle$ such that $f_{\mathrm{out}} \colon (\Phi S)^{n+1} \to (\Phi S)^m$ and $f_{\mathrm{fb}} \colon (\Phi S)^{n+1} \to \Phi S$ ,is defined by

$$\mathrm{Tr}^k_{n,m}(f) = f_{\mathrm{out}} \circ \langle \mathrm{id}_n, s(f_{\mathrm{fb}}) \rangle$$

where $s(f_{\mathrm{fb}})$ is a fixed point for $\mathrm{i} \circ \mathrm{next} \circ f_{\mathrm{fb}}$.

Since the trace of a chain map is a chain map, it is as well causal by Theorem 4.

## 5.2 Probabilistic Computations

Let us denote by $\mathcal{D}\colon \mathbf{Set} \to \mathbf{Set}$ the (functor of the) finite probability distribution monad. The elements of $\mathcal{D}(X)$ are maps $d\colon X \to [0,1]$ that have only finitely many elements in the support $\mathrm{supp}(d) = \{x \in X \mid d(x) \neq 0\}$ and such that $\sum_{x\in\mathrm{supp}(d)} d(x) = 1$. On maps $f\colon X \to Y$, $\mathcal{D}$ is defined by $\mathcal{D}(f)(d)(y) = \sum_{f(x)=y} d(x)$. We can now consider probabilistic stream systems, also known as labelled Markov chains, which are coalgebras for the composed functor $\mathcal{D}_R = \mathcal{D}(R \times \mathrm{Id})$.

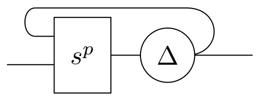



**Figure 2** Diagram for computing discounted sum $\mathrm{ds}_p$

Let us construct a discounted sum operation $\mathrm{ds}_p\colon \Phi\mathcal{D}_R \to \Phi\mathcal{D}_R$ for $p \in [0,1]$ as the diagram displayed in Figure 2. First of all, the convex sum induces a distributive law $c^p\colon \Sigma_2\mathcal{D}_R \to \mathcal{D}_R$ given by $c^p_X(d_1,d_2)(r,x,y) = pd_1(r,x) + (1-p)d_2(r,y)$. This gives us a causal map $\hat{c^p}\colon (\Phi\mathcal{D}_R)^2 \to \Phi\mathcal{D}_R$. Finally, we obtain $\mathrm{ds}_p$ as $\mathrm{Tr}(\Delta \circ \hat{c^p})$, where $\Delta$ is the diagonal map $\Phi\mathcal{D}_R \to (\Phi\mathcal{D}_R)^2$.

Note that $\hat{c^p}$ is *not* compatible with the initial value induced by the unit $\eta^{\mathcal{D}}$ of the distribution monad, which is defined by $\eta^{\mathcal{D}}_X(x) = 1$. In particular, we obtain $(s^p \circ \Sigma_2\eta^{\mathcal{D}})(x,y) = p\eta^{\mathcal{D}}(x) + (1-p)\eta^{\mathcal{D}}(y)$ and this is not a Dirac distribution given by $\eta^{\mathcal{D}}$, unless $x = y$.

## 5.3 Remark

A potential example that one could additionally consider is the category of presheaves $\mathrm{PSh(P)} = [P^{\mathrm{op}}, \mathbf{Set}]$ on a preordered set $P$. The category PSh(P) is Cartesian closed and for a limit preserving functor $F$, the carrier of a final coalgebra for F is a presheaf, which is a functor $\nu F\colon P^{\mathrm{op}} \to \mathbf{Set}$. Hence a causal morphism $f\colon \nu F \to \nu F$ is a natural transformation and the corresponding chain map is a morphism between a final chain, which is a diagram in $\overleftarrow{\mathrm{PSh(P)}} = [\alpha^{\mathrm{op}}, \mathrm{PSh(P)}] = [\alpha^{\mathrm{op}}, [P^{\mathrm{op}}, \mathbf{Set}]]$, for a limit ordinal $\alpha$. Moreover, PSh(P) has a generator. Therefore, one could investigate the meaning of causality using theorem 4 and theorem 8.

# 6 Summary, Related Work and Future Work

We have defined causal morphisms on the carrier of a final coalgebra $\nu F$ for a limit preserving endofunctor $F$ on arbitrary cartesian closed categories $\mathbf{C}$. We have seen, based on the construction of a final coalgebra via final chains, that there is a one-to-one correspondence between causal maps in $\mathrm{Caus}(\nu F, \nu F)$ and chain maps in $\overleftarrow{\mathbf{C}}(\Phi F, \Phi F)$, where $\nu F$ is isomorphic to the limit of $\Phi F$. For a locally small category with a generator, we equipped $\nu F$ with a metric and found out that causal morphisms are metric maps and vice versa. Additionally, we have constructed on a category of descending chains a (parameterised) traced symmetric monoidal category, on which causal morphisms (simply chain maps between final chains) are closed under sequential and parallel composition and under recursion via the trace operator.

We are well aware of the work of [16] and [14] which both give a definition of causal functions via finite approximations, but both work on **Set** and give the equivalence between causal functions on final coalgebras and morphisms on their finite approximations. We can easily extend our definition to causal algebras, as in [14], which gives us the inspiration to more general notion of causality. [16] introduced recursion in their work, which could be achieved in a traced symmetric monoidal category. They also defined linear causal maps, but for our case, it is enough to talk about linearity since we show that linear maps are causal.

For future work, we consider working on other cartesian closed categories such as $G - \mathbf{Set}$ of sets with group actions from $G$, particularly nominal set; and also on the CCC of quasi-Borel spaces on which one can formalize some probability theory. One could use monoidal closed categories instead of cartesian closed and see how everything works out. We would also like to extend the notion of causality to more general continuity properties.

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# A Complete Proof of Theorem 16

1. *Naturality on* $(\tau, n)$*:* $\mathrm{Tr}^k_{-,(\gamma,m)} \colon \mathbf{P}_\sigma(- \otimes \underline{k}, (\gamma, m) \otimes \underline{k}) \to \mathbf{P}_\sigma(-, (\gamma, m))$ is a natural transformation.
Let $f\colon (\tau, n) \otimes \underline{k} \to (\gamma, m) \otimes \underline{k}$ be $k$-feedback and $g\colon (\tau', n') \to (\tau, n)$, both morphisms in $\mathbf{P}_\sigma$. We need to show that

$$\mathrm{Tr}^k_{(\tau',n'),(\gamma,m)}(f \circ (g \otimes \mathrm{id}_{\underline{k}})) = \mathrm{Tr}^k_{(\tau,n),(\gamma,m)}(f) \circ g. \tag{11}$$

We first write down, using the definition, the left-hand side of equality (11). Since $f$ is $k$-feedback, then we have

$$(f \circ (g \otimes \mathrm{id}_{\underline{k}}))_{\mathrm{out}} = f_{\mathrm{out}} \circ (g \otimes \mathrm{id}_{\underline{k}}) \text{ and}$$
$$(f \circ (g \otimes \mathrm{id}_{\underline{k}}))_{\mathrm{fb}} = f_{\mathrm{fb}} \circ (g \otimes \mathrm{id}_{\underline{k}}).$$

Hence, by Equation (2), we have

$$\mathrm{Tr}^k_{(\tau',n'),(\gamma,m)}(f \circ (g \otimes \mathrm{id}_{\underline{k}})) = f_{\mathrm{out}} \circ (g \otimes \mathrm{id}_{\underline{k}}) \circ \langle \mathrm{id}_{(\tau',n')}, s(f_{\mathrm{fb}} \circ (g \otimes \mathrm{id}_{\underline{k}}))\rangle, \tag{12}$$

where $s(f_{\mathrm{fb}} \circ (g \otimes \mathrm{id}_{\underline{k}})) : \tau' \times \sigma^{n'} \to \sigma^k$ is a solution for $\hat{\mathrm{i}}^k \circ \mathrm{next}_{\sigma^k} \circ f_{\mathrm{fb}} \circ (g \otimes \mathrm{id}_{\underline{k}})$, and

$$s(f_{\mathrm{fb}} \circ (g \otimes \mathrm{id}_{\underline{k}})) = \hat{\mathrm{i}}^k \circ \mathrm{next}_{\sigma^k} \circ f_{\mathrm{fb}} \circ (g \otimes \mathrm{id}_{\underline{k}}) \circ \langle \mathrm{id}_{(\tau',n')}, s(f_{\mathrm{fb}} \circ (g \otimes \mathrm{id}_{\underline{k}}))\rangle.$$

The right hand side of (11) gives us

$$\mathrm{Tr}^k_{(\tau,n),(\gamma,m)}(f) \circ g = f_{\mathrm{out}} \circ \langle \mathrm{id}_{(\tau,n)}, s(f_{\mathrm{fb}})\rangle \circ g, \tag{13}$$

with $s(f_{\mathrm{fb}}) : \tau \times \sigma^n \to \sigma^k$ being the fixed point of $\hat{\mathrm{i}}^k \circ \mathrm{next}_{\sigma^k} \circ f_{\mathrm{fb}}$ and

$$s(f_{\mathrm{fb}}) = \hat{\mathrm{i}}^k \circ \mathrm{next}_{\sigma^k} \circ f_1 \circ \langle \mathrm{id}_{(\tau,n)}, s\rangle. \tag{14}$$

It remains to prove that,

$$f_1 \circ \langle \mathrm{id}_{(\tau,n)}, s^k\rangle \circ g = f_1 \circ (g \otimes \mathrm{id}_{\underline{k}}) \circ \langle \mathrm{id}_{\tau',n'}, r^k\rangle. \tag{15}$$

By Lemma 14, can replace $s(f_{\mathrm{fb}} \circ (g \otimes \mathrm{id}_{\underline{k}}))$ in (12) by $s(f_{\mathrm{fb}}) \circ g$. Then, we get

$$\begin{aligned}
\mathrm{Tr}^k_{(\tau',n'),(\gamma,m)}(f \circ (g \otimes \mathrm{id}_{\underline{k}})) &= f_{\mathrm{out}} \circ (g \otimes \mathrm{id}_{\underline{k}}) \circ \langle \mathrm{id}_{(\tau',n')}, s(f_{\mathrm{fb}}) \circ g\rangle, \\
&= f_{\mathrm{out}} \circ \langle g \circ \mathrm{id}_{(\tau',n')}, s(f_{\mathrm{fb}}) \circ g\rangle, \\
&= f_{\mathrm{out}} \circ \langle \mathrm{id}_{(\tau,n)} \circ g, s(f_{\mathrm{fb}}) \circ g\rangle, \\
&= f_{\mathrm{out}} \circ \langle \mathrm{id}_{(\tau,n)}, s(f_{\mathrm{fb}})\rangle \circ g, \\
&= \mathrm{Tr}^k_{(\tau,n),(\gamma,m)}(f) \circ g.
\end{aligned}$$

Hence, equality (11).
2. *Naturality on* $(\gamma, m)$*:* $\mathrm{Tr}^k_{(\tau,n),-} \colon \mathbf{P}_\sigma((\tau, n) \otimes \underline{k}, - \otimes \underline{k}) \to \mathbf{P}_\sigma((\tau, n), -)$ is a natural transformation.
Let $f\colon (\tau, n) \otimes \underline{k} \to (\gamma, m) \otimes \underline{k}$ and $g\colon (\gamma, m) \to (\gamma', m')$, we need to show that

$$\mathrm{Tr}^k_{(\tau,n),(\gamma',m')}((g \otimes \mathrm{id}_{\underline{k}}) \circ f) = g \circ \mathrm{Tr}^k_{(\tau,n),(\gamma,m)}(f). \tag{16}$$

For the $k$-feedback morphism $(g \otimes \mathrm{id}_{\underline{k}}) \circ f$,

$$((g \otimes \mathrm{id}_{\underline{k}}) \circ f)_{\mathrm{out}} = g \circ f_{\mathrm{out}}, \qquad \text{and}$$

$$((g \otimes \mathrm{id}_{\underline{k}}) \circ f)_{\mathrm{fb}} = f_{\mathrm{fb}}.$$

By definition, we then have:

$$\mathrm{Tr}^{k}_{(\tau,n),(\gamma',m')}((g \otimes \mathrm{id}_{\underline{k}}) \circ f) = g \circ f_{\mathrm{out}} \circ \langle \mathrm{id}_{(\tau,n)}, s(f_{\mathrm{fb}})\rangle. \tag{17}$$

We also have, on the other hand, by definition of $\mathrm{Tr}^{k}_{(\tau,n),(\gamma,m)}(f)$, that

$$g \circ \mathrm{Tr}^{k}_{(\tau,n),(\gamma,m)}(f) = g \circ f_{\mathrm{out}} \circ \langle \mathrm{id}_{(\tau,n)}, s(f_{\mathrm{fb}})\rangle. \tag{18}$$

Hence equality (4).

3. *Dinaturality on $\underline{k}$:* $\mathrm{Tr}^{-}_{(\tau,n),(\gamma,m)} \colon \mathbf{P}_\sigma((\tau,n) \otimes -, (\gamma,m) \otimes -) \to \mathbf{P}_\sigma((\tau,n),(\gamma,m))$ is a dinatural transformation, on the full subcategory $\mathbf{H}_\sigma$ with objects of the form $\underline{n} = (K\mathbf{1}, n)$ for all $n \in \mathbb{N}$, and if $\mathrm{i}_{\sigma^{\underline{k}}}$ at every $k \in \mathbb{N}$ satisfies for each $g \colon \underline{k} \to \underline{k}'$, $g \circ \hat{\mathrm{i}}^{k} = \hat{\mathrm{i}}^{k'} \circ \blacktriangleright g$.

Let $f \colon (\tau,n) \otimes \underline{k} \to (\gamma,m) \otimes \underline{k}'$ and $g \colon \underline{k}' \to \underline{k}$, we need to show that

$$\mathrm{Tr}^{k}_{(\tau,n),(\gamma,m)}((\mathrm{id}_{(\gamma,m)} \otimes g) \circ f) = \mathrm{Tr}^{k'}_{(\tau,n),(\gamma,m)}(f \circ (\mathrm{id}_{(\tau,n)} \otimes g)). \tag{19}$$

Note that $(\mathrm{id}_{(\gamma,m)} \otimes g) \circ f$ is $k$-feedback with

$$((\mathrm{id}_{(\gamma,m)} \otimes g) \circ f)_{\mathrm{out}} = f_{\mathrm{out}},$$
$$((\mathrm{id}_{(\gamma,m)} \otimes g) \circ f)_{\mathrm{fb}} = (g \circ f)_{\mathrm{fb}},$$

and $f \circ (\mathrm{id}_{(\tau,n)} \otimes g)$ is $k'$-feedback, with

$$(f \circ (\mathrm{id}_{(\tau,n)} \otimes g))_{\mathrm{out}} = f_{\mathrm{out}} \circ (\mathrm{id}_{(\tau,n)} \otimes g), \qquad \text{and}$$
$$(f \circ (\mathrm{id}_{(\tau,n)} \otimes g))_{\mathrm{fb}} = f_{\mathrm{fb}} \circ (\mathrm{id}_{(\tau,n)} \otimes g);$$

such that $f_{\mathrm{out}} \colon \tau \times \sigma^n \times \sigma^k \to \gamma \times \sigma^m$ and $f_{\mathrm{fb}} \colon \tau \times \sigma^n \times \sigma^k \to \sigma^{k'}$.
Then, by Theorem 16, we have

$$\mathrm{Tr}^{k}_{(\tau,n),(\gamma,m)}((\mathrm{id}_{(\gamma,m)} \otimes g) \circ f) = f_{\mathrm{out}} \circ \langle \mathrm{id}_{(\tau,n)}, s(g \circ f_{k'})\rangle; \tag{20}$$

and

$$\begin{aligned}\mathrm{Tr}^{k'}_{(\tau,n),(\gamma,m)}(f \circ (\mathrm{id}_{(\tau,n)} \otimes g)) &= f_{\mathrm{out}} \circ (\mathrm{id}_{(\tau,n)} \otimes g) \circ \langle \mathrm{id}_{(\tau,n)}, s(f_{k'} \circ (\mathrm{id}_{(\tau,n)} \otimes g))\rangle,\\ &= f_{\mathrm{out}} \circ \langle \mathrm{id}_{(\tau,n)}, g \circ s(f_{k'} \circ (\mathrm{id}_{(\tau,n)} \otimes g))\rangle.\end{aligned}$$

Let $s^{k'} = s(f_{k'} \circ (\mathrm{id}_{(\tau,n)} \otimes g))$, a solution for $\mathrm{i}_{\sigma_{k'}} \circ \mathrm{next}_{\sigma^{k'}} \circ f_{k'} \circ (\mathrm{id}_{(\tau,n)} \otimes g)$, then by Lemma 15, $g \circ s^{k'}$ is a solution for $\hat{\mathrm{i}}^{k} \circ \mathrm{next}_{\sigma^k} \circ g \circ f_{k'}$. Hence, we can substitute $s(g \circ f_{k'})$ in (20), by $g \circ s^{k'}$, and we get

$$\begin{aligned}\mathrm{Tr}^{k}_{(\tau,n),(\gamma,m)}((\mathrm{id}_{(\gamma,m)} \otimes g) \circ f) &= f_{\mathrm{out}} \circ \langle \mathrm{id}_{(\tau,n)}, s(g \circ f_{k'})\rangle,\\ &= f_{\mathrm{out}} \circ \langle \mathrm{id}_{(\tau,n)}, g \circ s^{k'}\rangle,\\ &= \mathrm{Tr}^{k'}_{(\tau,n),(\gamma,m)}(f \circ (\mathrm{id}_{(\tau,n)} \otimes g)).\end{aligned}$$

▶ Remark 20. In the case where we do not have $g \circ \mathrm{i}_{\sigma^{\underline{k}}} = \mathrm{i}_{\sigma^{\underline{k}'}} \circ \blacktriangleright g$, dinaturality is not satified.
We have now seen that trace in Theorem 16 is a family of natural morphisms, we are left to check if they fulfill the three axioms of trace in [9], for symmetric monoidal categories.

4. *Vanishing 1:* Let $f\colon (\tau,n)\otimes \underline{0} \to (\gamma,m)\otimes \underline{0}$ and $\iota_r\colon -\otimes \underline{1} \to -$, where $\iota_r$ is the right unitor. Then we need to show, that

$$\mathrm{Tr}^0_{(\tau,n),(\gamma,m)}(f) = \iota_{r(\gamma,m)} \circ f \circ \iota^{-1}_{r(\tau,n)}. \tag{21}$$

Note that $\mathrm{Tr}^0_{(\tau,n),(\gamma,m)}\colon \mathbf{P}_\sigma((\tau,n),(\gamma,m)) \to \mathbf{P}_\sigma((\tau,n),(\gamma,m))$
In this case, $f$ is 0-feedback, therefore $f_{\mathrm{out}} = f$. Hence

$$\begin{aligned}\mathrm{Tr}^0_{(\tau,n),(\gamma,m)}(f) &= f\\ &= \iota_{r(\gamma,m)} \circ f \circ \iota^{-1}_{r(\tau,n)}.\end{aligned}$$

5. *Vanishing 2:* Let $f : (\tau,n)\otimes \underline{1}\otimes \underline{1} \to (\gamma,m)\otimes \underline{1}\otimes \underline{1}$ We need to show that

$$\mathrm{Tr}^2_{(\tau,n),(\gamma,m)}(f) = \mathrm{Tr}^1_{(\tau,n),(\gamma,m)}(\mathrm{Tr}^1_{(\tau,n+1),(\gamma,m+1)}(f)) \tag{22}$$

$f$ is a 2-feedback, and we have $\underline{2} = \underline{1}\otimes\underline{1} = \sigma^{1+1} = \sigma^2 \cong \sigma\times\sigma$. We shall decompose $f$ as follows, for us to be able to unfold de definition.
$f = \langle f_{\mathrm{out}}, f_2\rangle = \langle f_{\mathrm{out}}, f_{21}, f_1\rangle = \langle f_{\mathrm{out,2out}}, f_1\rangle$ such that

$$\begin{aligned} f\colon \tau\times\sigma^n\times\sigma\times\sigma &\to \gamma\times\sigma^m\times\sigma\times\sigma\\ f_{\mathrm{out}}\colon \tau\times\sigma^n\times\sigma\times\sigma &\to \gamma\times\sigma^m\\ f_{\mathrm{out,2out}}\colon \tau\times\sigma^n\times\sigma\times\sigma &\to \gamma\times\sigma^m\times\sigma\\ f_{21}\colon \tau\times\sigma^n\times\sigma\times\sigma &\to \sigma\\ f_1\colon \tau\times\sigma^n\times\sigma\times\sigma &\to \sigma\end{aligned}$$

Let us first unfold the definition of the right hand side of equation (22).

$$\mathrm{Tr}^1_{(\tau,n+1),(\gamma,n+1)}(f) = f_1\circ\langle \mathrm{id}_{(\tau,n+1)}, s_1\rangle\ , \tag{23}$$

With $s_1 : \tau\times\sigma^{n+1}\to\sigma$ being a solution for $\hat{\imath}^1\circ\mathrm{next}_\sigma\circ f_1$ and
$s_1 = \hat{\imath}^1\circ\mathrm{next}_\sigma\circ f_1\circ\langle \mathrm{id}_{(\tau,n+1)}, s_1\rangle$ . Then

$$\mathrm{Tr}^1_{(\tau,n),(\gamma,m)}(\mathrm{Tr}^1_{(\tau,n+1),(\gamma,m+1)}(f)) = (f_1\circ\langle \mathrm{id}_{(\tau,n+1)}, s_1\rangle)_1\circ\langle \mathrm{id}_{(\tau,n)}, s_2\rangle, \tag{24}$$

such that $s_2 : \tau\times\sigma^n\to\sigma$ is a solution for $\hat{\imath}^1\circ\mathrm{next}_\sigma\circ(f_{\mathrm{out,2out}}\circ\langle \mathrm{id}_{(\tau,n+1)}, s_1\rangle)_2$ and
$s_2 = \hat{\imath}^1\circ\mathrm{next}_\sigma\circ f_{21}\circ\langle \mathrm{id}_{(\tau,n+1)}, s_1\rangle\circ\langle \mathrm{id}_{(\tau,n)}, s_2\rangle$, where

$$\begin{aligned}(f_{\mathrm{out,2out}}\circ\langle \mathrm{id}_{(\tau,n+1)}, s_1\rangle)_1 &= f_{\mathrm{out}}\circ\langle \mathrm{id}_{(\tau,n+1)}\ s_1\rangle \qquad \text{and}\\ (f_{\mathrm{out,2out}}\circ\langle \mathrm{id}_{(\tau,n+1)}, s_1\rangle)_2 &= f_{21}\circ\langle \mathrm{id}_{(\tau,n+1)}, s_1\rangle \qquad .\end{aligned}$$

Hence

$$\mathrm{Tr}^1_{(\tau,n),(\gamma,m)}(\mathrm{Tr}^1_{(\tau,n+1),(\gamma,m+1)}(f)) = f_{\mathrm{out}}\circ\langle \mathrm{id}_{(\tau,n+1)}, s_1\rangle\circ\langle \mathrm{id}_{(\tau,n)}, s_2\rangle. \tag{25}$$

Now we can proceed to the left-hand side of equation (22). We have $f = \langle f_{\mathrm{out}}, \langle f_{21}, f_1\rangle\rangle$, and

$$\mathrm{Tr}^2_{(\tau,n),(\gamma,m)}(f) = f_{\mathrm{out}}\circ\langle \mathrm{id}_{(\tau,n)}, s\rangle, \tag{26}$$

where $s : \tau\times\sigma^n\to\sigma\times\sigma$ is a solution for $\hat{\imath}^2\circ\mathrm{next}_{\sigma^2}\circ\langle f_{21}, f_1\rangle : \tau\times\sigma^n\times\sigma\times\sigma\to\sigma\times\sigma$.
We can show that $t = \langle s_2, s_1\circ\langle \mathrm{id}_{(\tau,n)}, s_2\rangle\rangle$ is a solution for $\hat{\imath}^2\circ\mathrm{next}_{\sigma^2}\circ\langle f_{21}, f_1\rangle$, ie

$$t = \hat{\imath}^2\circ\mathrm{next}_{\sigma^2}\circ\langle f_{21}, f_1\rangle\circ\langle \mathrm{id}_{(\tau,n)}, t\rangle. \tag{27}$$

We have the following identities

$$\begin{aligned}\langle \mathrm{id}_{(\tau,n+1)}, s_1\rangle \circ \langle \mathrm{id}_{(\tau,n)}, s_2\rangle =& \langle \mathrm{id}_{(\tau,n)}, s_2, s_1 \circ \langle \mathrm{id}_{(\tau,n)}, s_2\rangle\rangle \\ =& \langle \mathrm{id}_{(\tau,n)}, t\rangle.\end{aligned}$$

and

$$\begin{aligned}t &= \langle s_2, s_1 \circ \langle \mathrm{id}_{(\tau,n)}, s_2\rangle\rangle, \\ &= \langle s_2, \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1 \circ \langle \mathrm{id}_{(\tau,n+1)}, s_1\rangle \circ \langle \mathrm{id}_{(\tau,n)}, s_2\rangle\rangle, \\ &= \langle s_2, \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1 \circ \langle \mathrm{id}_{(\tau,n)}, t\rangle\rangle, \\ &= \langle \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_{21} \circ \langle \mathrm{id}_{(\tau,n+1)}, s_1\rangle \circ \langle \mathrm{id}_{(\tau,n)}, s_2\rangle, \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1 \circ \langle \mathrm{id}_{(\tau,n)}, t\rangle\rangle, \\ &= \langle \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_{21} \circ \langle \mathrm{id}_{(\tau,n)}, t\rangle, \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1 \circ \langle \mathrm{id}_{(\tau,n)}, t\rangle\rangle, \\ &= \langle \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_{21}, \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1\rangle \circ \langle \mathrm{id}_{(\tau,n)}, t\rangle, \\ &= \hat{\mathrm{i}}^2 \circ \mathrm{next}_{\sigma^2} \circ \langle f_{21}, f_1\rangle \circ \langle\langle \mathrm{id}_{(\tau,n)}, t\rangle.\end{aligned}$$

Hence $t$ is a solution $\hat{\mathrm{i}}^2 \circ \mathrm{next}_{\sigma^2} \circ \langle f_{21}, f_1\rangle$. Therefore we have the following.

$$\begin{aligned}\mathrm{Tr}^2_{(\tau,n),(\gamma,m)}(f) &= f_{\mathrm{out}} \circ \langle \mathrm{id}_{(\tau,n)}, t\rangle \\ &= f_{\mathrm{out}} \circ \langle \mathrm{id}_{(\tau,n)}, s_2, s_1 \circ \langle \mathrm{id}_{(\tau,n)}, s_2\rangle\rangle \\ &= \mathrm{Tr}^1_{(\tau,n),(\gamma,m)}(\mathrm{Tr}^1_{(\tau,n+1),(\gamma,m+1)}(f))\end{aligned}$$

**6.** *Superposing:* Let $f\colon (\tau, n) \otimes \underline{1} \to (\gamma, m) \otimes \underline{1}$ and $g\colon (\tau', n') \to (\gamma', m')$, we need to show that

$$g \circ \mathrm{Tr}^1_{(\tau,n),(\gamma,m)}(f) = \mathrm{Tr}^1_{(\tau',n')\otimes(\tau,n),(\gamma',m')\otimes(\gamma,m)}(g \otimes f)\ . \tag{28}$$

We have

$$\begin{aligned}\mathrm{Tr}^1_{(\tau',n')\otimes(\tau,n),(\gamma',m')\otimes(\gamma,m)}(g \otimes f) &= (g \otimes f)_1 \circ \langle \mathrm{id}_{(\tau',n')\otimes(\tau,n)}, s\rangle \\ &= (g \otimes f_1) \circ \langle \mathrm{id}_{(\tau',n')\otimes(\tau,n)}, s\rangle\end{aligned}$$

where $s\colon \tau' \times \sigma^{n'} \times \tau \times \sigma^n \to \sigma$ is a solution for $\hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ (g \otimes f)_2 = \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1 \circ \pi_{(\tau,n+1)}$, i.e.

$$s = \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1 \circ \pi_{(\tau,n+1)} \circ \langle \mathrm{id}_{(\tau',n')\otimes(\tau,n)}, s\rangle, \tag{29}$$

$$\quad = \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1 \circ \langle \pi_{(\tau,n)} \circ \mathrm{id}_{(\tau',n')\otimes(\tau,n)}, s\rangle. \tag{30}$$

If $s(f_1)\colon \tau \times \sigma^n \to \sigma$ is a solution for $\hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1$, i.e.

$$s(f_1) = \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1 \circ \langle \mathrm{id}_{(\tau,n)}, s(f_1)\rangle\ .$$

then $s(f_1) \circ \pi_{(\tau,n)}\colon \tau' \times \sigma^{n'} \times \tau \times \sigma^n \to \sigma$, is a solution for $\hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1 \circ \pi_{(\tau,n+1)}$, because of the following

$$\begin{aligned}s(f_1) \circ \pi_{(\tau,n)} &= \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1 \circ \langle \mathrm{id}_{(\tau,n)}, s(f_1)\rangle \circ \pi_{(\tau,n)}, \\ &= \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1 \circ \langle \mathrm{id}_{(\tau,n)} \circ \pi_{(\tau,n)}, s(f_1) \circ \pi_{(\tau,n)}\rangle; \\ &= \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1 \circ \langle \pi_{(\tau,n)}, s(f_1) \circ \pi_{(\tau,n)}\rangle; \\ &= \hat{\mathrm{i}}^1 \circ \mathrm{next}_\sigma \circ f_1 \circ \langle \pi_{(\tau,n)} \circ \mathrm{id}_{(\tau',n')\otimes(\tau,n)}, s(f_1) \circ \pi_{(\tau,n)}\rangle.\end{aligned}$$

The last equality is similar to (30), by susbstituting $s(f_1) \circ \pi_{(\tau,n)}$ for $s$. Since, we have by definition

$$\mathrm{Tr}^1_{(\tau,n),(\gamma,m)}(f) = f_1 \circ \langle \mathrm{id}_{(\tau,n)}, s(f_1)\rangle \; . \tag{31}$$

$$\begin{aligned}
\mathrm{Tr}^1_{(\tau',n')\otimes(\tau,n),(\gamma',m')\otimes(\gamma,m)}(g \otimes f) &= (g \otimes f_1) \circ \langle \mathrm{id}_{(\tau',n')\otimes(\tau,n)}, s(f_1) \circ \pi_{(\tau,n)}\rangle \\
&= (g \otimes f_1) \circ \langle \mathrm{id}_{(\tau',n')} \otimes id_{(\tau,n)}, s(f_1) \circ \pi_{(\tau,n)}\rangle \\
&= (g \otimes f_1) \circ (id_{(\tau',n')} \otimes \langle \mathrm{id}_{(\tau,n)}, s(f_1)\rangle) \\
&= (id_{(\tau',n')} \circ g) \otimes (f_1 \circ \langle \mathrm{id}_{(\tau,n)}, s(f_1)\rangle), \text{ by bifunctoriality of } \otimes \\
&= g \otimes (f_1 \circ \langle \mathrm{id}_{(\tau,n)}, s(f_1)\rangle) \\
&= g \otimes \mathrm{Tr}^1_{(\tau,n),(\gamma,m)}(f)
\end{aligned}$$

Therefore, we have (28), which proves the superposition axiom.

**7.** *Yanking:* We need to show, for the component at $(\underline{1}, \underline{1})$ of the braiding, i.e. $\xi_{\underline{1},\underline{1}}$, that

$$\mathrm{Tr}^1_{(\underline{1},\underline{1})}(\xi_{\underline{1},\underline{1}}) = id_{\underline{1}}. \tag{32}$$

Note that $\xi_{\underline{1},\underline{1}} = \langle \pi_1, \pi_2\rangle$ , $\mathrm{Tr}^1_{(\underline{1},\underline{1})}(\xi_{\underline{1},\underline{1}}) = \pi_1 \circ \langle \mathrm{id}_{\underline{1}}, s(\pi_2)\rangle$ , where $s(\pi_2)\colon \sigma \to \sigma$ is a solution for $\pi_2$ . $id_{\underline{1}}$ is a solution for $\pi_2$ . Therefore,

$$\begin{aligned}
\mathrm{Tr}^1_{(\underline{1},\underline{1})}(\xi_{\underline{1},\underline{1}}) &= \pi_1 \circ \langle \mathrm{id}_{\underline{1}}, s\rangle \\
&= \pi_1 \circ \langle \mathrm{id}_{\underline{1}}, id_{\underline{1}}\rangle \\
&= id_{\underline{1}}
\end{aligned}$$

The dinaturality of $\mathrm{Tr}^{-}_{(\tau,n),(\gamma,m)}$ is only on $\mathbf{P}_\sigma$, and that is only fulfilled if for any $g \in \overleftarrow{\mathbf{C}}(\underline{k}, \underline{k})$, $\hat{\imath}^k \circ \blacktriangleright g = g \circ \hat{\imath}^{k'}$ .