Master d'Informatique Fondamentale – 2019
Semantics and Verification (M1)

TD de Sémantique et Vérification

VII– Generalised Büchi Automata and Linear Time Logic

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In this set of exercises, we will discuss properties of generalised NBAs and of LTL formulas.

Recommendation: The exercises are all purely pen and paper exercises. However, it is quite fun to implement notions from the course and the exercises. At the very end, you may obtain this way your very own model checker. This week, you may implement generalised non-deterministic Büchi automata, the product construction and model checking through checking the persistence property. Moreover, you may implement the syntax of LTL formulas. Note that your implementation will not be evaluated as part of the course.

Model Checking of ω-Regular Properties

Exercise 1.

Prove the verification theorem for ω-regular properties: Let TS be a finite transition system without terminal states over AP, P an ω-regular property over AP, and $\mathcal{A}$ a non-blocking NBA with alphabet $\mathcal{P}(\text{AP})$ and language $\mathcal{L}_\omega(\mathcal{A}) = P^{\complement}$. Then, the following statements are equivalent:

- $TS \models P$
- $\text{Traces}^\omega(TS) \cap \mathcal{L}_\omega(\mathcal{A}) = \emptyset$
- $TS \otimes \mathcal{A} \models P_{\text{pers}(\mathcal{A})}$, where $P_{\text{pers}(\mathcal{A})}$ is the persistence property "eventually forever $\neg F$" over the states of $\mathcal{A}$ seen as atomic propositions and where F are the final states of $\mathcal{A}$.

Generalised Non-Deterministic Büchi Automata

Exercise 2.

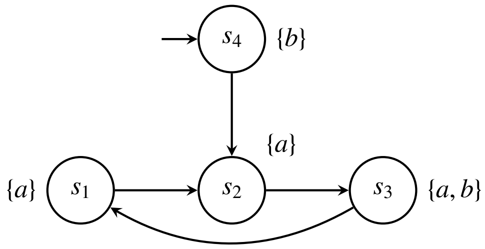
Let $\mathcal{G}$ be a generalised NBA. In the lecture, the construction of an NBA $\mathcal{A}$ from $\mathcal{G}$ was given.

1. Let $|\mathcal{A}|$ (resp. $|\mathcal{G}|$) be the cardinality of the states and transitions of $\mathcal{A}$ (resp. $\mathcal{G}$). Show that we have $|\mathcal{A}| = O(|\mathcal{G}| \cdot |F_{\mathcal{G}}|)$, where $F_{\mathcal{G}}$ is the acceptance set of $\mathcal{G}$.
2. Show that $\mathcal{L}_\omega(\mathcal{G}) = \mathcal{L}_\omega(\mathcal{A})$.

Linear Time Logic

Exercise 3.

Consider the following transition system over the set of atomic propositions $\{a, b\}$:



Indicate for each of the following LTL formulae the set of states for which these formulae are fulfilled:

A. $\bigcirc a$	C. $\Box b$	E. $\Box(b\ \mathcal{U}\ a)$
B. $\bigcirc\bigcirc\bigcirc a$	D. $\Box\Diamond a$	F. $\Diamond(a\ \mathcal{U}\ b)$

Exercise 4.

Prove the following equivalences of LTL formulas.

1. $\neg\bigcirc\varphi \equiv \bigcirc\neg\varphi$
2. $\neg\Box\varphi \equiv \Diamond\neg\varphi$
3. $\Box\Box\varphi \equiv \Box\varphi$
4. $\varphi\ \mathcal{U}\ \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi\ \mathcal{U}\ \psi))$
5. $\Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi$

Exercise 5.

Which of the following equivalences are correct? Prove the equivalence or provide a counterexample that illustrates that the formula on the left and the formula on the right are not equivalent.

A. $\Box\varphi \to \Diamond\psi \equiv \varphi\ \mathcal{U}\ (\psi \vee \neg\varphi)$

B. $\Diamond\Box\varphi \to \Box\Diamond\psi \equiv \Box(\varphi\ \mathcal{U}\ (\psi \vee \neg\varphi))$

C. $\Box\Box(\varphi \vee \neg\psi) \equiv \neg\Diamond(\neg\varphi \wedge \psi)$

D. $\Diamond(\varphi \wedge \psi) \equiv \Diamond\varphi \wedge \Diamond\psi$

E. $\Box\varphi \wedge \bigcirc\Diamond\varphi \equiv \Box\varphi$

F. $\Diamond\varphi \wedge \bigcirc\Box\varphi \equiv \Diamond\varphi$

G. $\Box\Diamond\varphi \to \Box\Diamond\psi \equiv \Box(\varphi \to \Diamond\psi)$

H. $\bigcirc\Diamond\varphi \equiv \Diamond\bigcirc\varphi$

I. $(\Diamond\Box\varphi) \wedge (\Diamond\Box\psi) \equiv \Diamond(\Box\varphi \wedge \Box\psi)$

Order-Theoretic Fixed Points

Let in the following $(S, \leq)$ be a poset. Recall that S is a complete lattice if all joins and meets exist in S. Suppose $f\colon S \to S$ is a map on S. We say that $x \in S$ is a pre-fixpoint (resp. post-fixpoint) if $f(x) \leq x$ (resp. $x \leq f(x)$). A least fixed point $x \in S$ of f is the least pre-fixpoint, that is, x is a pre-fixpoint of f and for any other pre-fixpoint y of f, we have $x \leq y$. Greatest fixed points are defined dually in terms of post-fixpoints.

Exercise 6.

Let $(S, \leq)$ be a poset and $f\colon S \to S$ a monotone map.

1. Show that if f has a least fixed point, then this fixed point is unique.
2. Suppose that S is a complete lattice. Show that f has a least fixed point.

Exercise 7.

Recall that $(L, \subseteq)$ with $L = \mathcal{P}(\mathcal{P}(\mathrm{AP})^\omega)$ is a complete lattice. We define the derivative of $\sigma \in \mathcal{P}(\mathrm{AP})^\omega$ by $\sigma' = \sigma_1\sigma_2\cdots$, and for $P \in L$ by $N(P) = \{\sigma \mid \sigma' \in P\}$. Let φ and ψ be LTL formulas and define $f\colon L \to L$ by

$$f(P) = [\![\psi]\!] \cup ([\![\varphi]\!] \cap N(P)),$$

where $[\![\psi]\!] = \{\sigma \mid \sigma \vDash \psi\}$ and analogous for φ. This map has a least fixed point by the previous question. Show that $[\![\varphi\ \mathcal{U}\ \psi]\!]$ is this least fixed point.