ENS DE LYON

TD de Sémantique et Vérification

Homework

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In this set of homework exercises, you will meet again transition systems, linear time properties, order theory and topology.

Instructions: Send your solutions as readable scans to me (Henning) or leave your copy in my mailbox in the printer room on the 3rd floor latest by Friday 12th April 2019, 14:00. Next to the questions, you find in a box the marks awarded to that question. In total you may earn 40 marks that contribute to 50% of your final grade. You may work together on the exercises, but it is expected that you provide your own worked out solutions.

Transition Systems

Let TS be a transition system with $TS = (S, \mathrm{Act}, \to, I, \mathrm{AP}, L)$. We denote for $s \in S$ and $A \subseteq \mathrm{AP}$ we let

$$s^{(A)} = \{s' \mid \exists \alpha \in \mathrm{Act}.\, s \xrightarrow{\alpha} s' \text{ and } L(s') = A\}$$

and call $s^{(A)}$ the *A-successors of s*. The transition system TS is called AP-*deterministic* if $|I| \leq 1$ and every state has at most one A-successor for all A, that is, if

$$\forall s \in S.\, \forall A \subseteq \mathrm{AP}.\, \left|s^{(A)}\right| \leq 1.$$

Important: We assume that transition systems have no final states.

Exercise 1.

Let T and T' be transition systems with the same set of atomic propositions AP. Prove the following relationship between trace inclusion and finite trace inclusion:

$\boxed{2}$ 1. For AP-deterministic T and T' :

$$\mathrm{Traces}^{\omega}(T) = \mathrm{Traces}^{\omega}(T') \text{ if and only if } \mathrm{Traces}_{\mathrm{fin}}(T) = \mathrm{Traces}_{\mathrm{fin}}(T').$$

$\boxed{2}$ 2. Give concrete examples of T and T' where at least one of them is not AP-deterministic, but

$$\mathrm{Traces}^{\omega}(T) \nsubseteq \mathrm{Traces}^{\omega}(T') \text{ and } \mathrm{Traces}_{\mathrm{fin}}(T) = \mathrm{Traces}_{\mathrm{fin}}(T').$$

Safety Properties

Exercise 2.

$\boxed{1}$ 1. Show that for any transition system T, the set $\mathrm{cl}(\mathrm{Traces}^{\omega}(TS))$ is a safety property such that $T \vDash \mathrm{cl}(\mathrm{Traces}^{\omega}(T))$.

$\boxed{3}$ 2. Show that safety properties are not closed under arbitrary unions To this end, give a family of safety properties $\{P_i\}_{i \in I}$ for some set I, such that $\bigcup_{i \in I} P_i$ is not a safety property. *(Hint: Use that singleton sets are closed and that closed sets contain all limit points, see TD4.)*

Order Theory

Exercise 3.

In the following, we assume that S and T are posets. Moreover, we denote by $\sqsubseteq_S$ the point-wise order on maps into S, that is, for $h, k \colon X \to S$ we have $h \sqsubseteq_S k$ if $\forall x \in X.\, h(x) \leq_S k(x)$.

$\boxed{1}$ 1. Suppose $f \colon S \to T$ is a monotone function and $X \subseteq S$. Prove that if $\bigvee X$ and $\bigvee f(X)$ exist, then $\bigvee f(X) \leq f(\bigvee X)$.

$\boxed{1}$ 2. Let $f \colon S \to T$ and $g \colon T \to S$ be monotone functions. Show that (f, g) is a Galois connection if and only if $f \circ g \sqsubseteq_S \mathrm{id}$ and $\mathrm{id} \sqsubseteq_T g \circ f$.

$\boxed{2}$ 3. Let $f \dashv g : S \leftrightarrow T$ and $f' \dashv g : S \leftrightarrow T$ be Galois connections. Show that $f = f'$.

Exercise 4.

Let S and T be posets and assume that S has all meets, that is, for all sets $X \subseteq S$ the meet $\bigwedge X$ exists in S. Suppose that $g \colon S \to T$ is a monotone map that preserves meets, that is, $\forall X \subseteq S.\, g(\bigwedge X) = \bigwedge g(X)$.

$\boxed{2}$ 1. Show that g has a left-adjoint.

Exercise 5.

Let S be a poset that has all meets. Given a set I, we denote by S^I the set of all function $I \to S$ (note that I has no order relation). The set S^I is a poset with the point-wise order $\sqsubseteq_S$. We define the constant map $K_I \colon S \to S^I$ by $K_I(x)(i) = x$, which is clearly monotone.

$\boxed{1}$ 1. Show that K preserves meets. This gives by the previous exercise that K has a left-adjoint L.

$\boxed{1}$ 2. Show that any left-adjoint L of K maps a family $\rho \colon I \to S$ to the join $\bigvee\{\rho(i) \mid i \in I\}$.

$\boxed{1}$ 3. Conclude that S is a complete lattice, i.e., has also all joins.

Exercise 6.

Let Σ be a set. We define the *prefix order* on Σ^* as follows. Given $w, v \in \Sigma^*$, write $w \leq v$ if w is a prefix of v. Moreover, we define the *completion* $\widehat{\Sigma^*}$ of Σ^* by $\widehat{\Sigma^*} = \Sigma^* \cup \{\top\}$, and an order $\preceq$ on it by: $w \preceq v$ iff $w \leq v$ for $w, v \in \Sigma^*$ and $x \preceq \top$ for $x \in \widehat{\Sigma^*}$.

$\boxed{1}$ 1. Show that $(\Sigma^*, \leq)$ is a poset.

$\boxed{2}$ 2. Show that $(\widehat{\Sigma^*}, \preceq)$ is a poset.

$\boxed{2}$ 3. Show that $\widehat{\Sigma^*}$ has binary joins, that is, all $x, y \in \widehat{\Sigma^*}$ have a least upper bound $w \sqcup v$.

$\boxed{2}$ 4. Let $\mathrm{ext} \colon \widehat{\Sigma^*} \to \mathcal{P}(\Sigma^\omega)$ be given as follows.

$$\begin{aligned} \mathrm{ext}(w) &= w \cdot \Sigma^\omega \qquad \text{for } w \in \Sigma^* \\ \mathrm{ext}(\top) &= \emptyset \end{aligned}$$

Show that ext turns joins into meets, that is, show that $\mathrm{ext}(x \sqcup y) = \mathrm{ext}(x) \cap \mathrm{ext}(y)$.

Topology

Let $(X, \mathcal{U})$ be a topological space. A *cover* of X is a collection C of open sets in X, such that $\bigcup C = X$. We say that X is *compact* if for every cover C there is a finite subcover C' with $C' \subseteq C$.

Exercise 7.

Let Σ be a finite, non-empty set. Recall the metric d on Σ^ω from TD4 with

$$d(\sigma, \tau) = \begin{cases} 0, & \sigma = \tau \\ 2^{-\min\{k \in \mathbb{N} \mid \sigma(k) \neq \tau(k)\}}, & \sigma \neq \tau \end{cases}$$

and let $\mathcal{U}_d$ be given by $\mathcal{U}_d = \{U \subseteq \Sigma^\omega \mid \forall \sigma \in U.\, \exists \varepsilon > 0.\, B_\varepsilon(\sigma) \subseteq U\}$ as before.

1 1. Let $\mathcal{B} = \{\mathrm{ext}(x) \mid x \in \widehat{\Sigma^*}\}$ and $\mathcal{U} = \{\bigcup O \mid O \subseteq \mathcal{B}\}$. Show that $\mathcal{B}$ is closed under intersection.

1 2. Show that $\mathcal{U} = \mathcal{U}_d$.

2 3. Show that Σ^ω is compact.

Exercise 8.

(Topological decomposition theorem). Recall from TD4 that a set is dense if its closure is the whole space X.

1 1. Show that $D \subseteq X$ is dense if and only if for every $x \in X$ and $O \in \mathcal{N}_x$ that $D \cap O \neq \emptyset$.

3 2. Let $S \subseteq X$ be any subset of X. Show that there are a closed set C and a dense set D, such that, $S = C \cap D$.

................................32 marks breaking-point for a grade of 16 ....

Let $(X, \mathcal{U})$ be a topological space.

- Given a subset Y of X, the *topology induced on* Y is the collection $\{U \cap Y \mid U \in \mathcal{U}\}$ (this is indeed a topology).
- Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that a sequence $(y_n)_{n\in\mathbb{N}}$ is a *subsequence* of $(x_n)_{n\in\mathbb{N}}$ if there is a sequence $(i_n)_{n\in\mathbb{N}}$, such that for all $n \in \mathbb{N}$, $y_n = x_{i_n}$.

Exercise 9.

2 1. Let X be a compact topological space and $(C_n)_{n\in\mathbb{N}}$ a decreasing sequence of non-empty closed sets, that is, for all $n \in \mathbb{N}$, we have $C_n \supseteq C_{n+1}$ and $C_n \neq \emptyset$. Show that $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$.

2 2. Let X be compact and $Y \subseteq X$ closed. Show that Y is compact with the induced topology.

4 3. Show that if a metric space (X, d) is compact with its generated topology, then X is sequentially compact, that is, show that every sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence. (*Note: The other direction does not work with sequences for general topological spaces, but requires their generalisation to nets or filters.*)